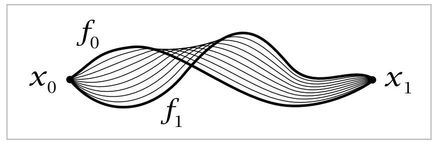
Paths and Homotopy

The fundamental group will be defined in terms of loops and deformations of loops. Sometimes it will be useful to consider more generally paths and their deformations, so we begin with this slight extra generality.

By a **path** in a space X we mean a continuous map $f : I \to X$ where I is the unit interval $[0,1]$. The idea of continuously deforming a path, keeping its endpoints fixed, is made precise by the following definition. A **homotopy** of paths in X is a family $f_t : I \to X$, $0 \le t \le 1$, such that

(1) The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.

(2) The associated map $F : I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous.



When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be **homotopic**. The notation for this is $f_0 \simeq f_1$.

Example 1.1: Linear Homotopies. Any two paths f_0 and f_1 in $\mathbb{R}^n$ having the same endpoints x_0 and x_1 are homotopic via the homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$. During this homotopy each point $f_0(s)$ travels along the line segment to $f_1(s)$ at constant speed. This is because the line through $f_0(s)$ and $f_1(s)$ is linearly parametrized as $f_0(s) + t[f_1(s) - f_0(s)] = (1-t)f_0(s) + tf_1(s)$, with the segment from $f_0(s)$ to $f_1(s)$ covered by t values in the interval from 0 to 1. If $f_1(s)$ happens to equal $f_0(s)$ then this segment degenerates to a point and $f_t(s) = f_0(s)$ for all t. This occurs in particular for $s = 0$ and $s = 1$, so each f_t is a path from x_0 to x_1. Continuity of the homotopy f_t as a map $I \times I \to \mathbb{R}^n$ follows from continuity of f_0 and f_1 since the algebraic operations of vector addition and scalar multiplication in the formula for f_t are continuous.

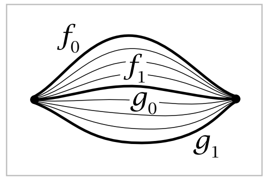
This construction shows more generally that for a convex subspace $X \subset \mathbb{R}^n$, all paths in X with given endpoints x_0 and x_1 are homotopic, since if f_0 and f_1 lie in X then so does the homotopy f_t.

Before proceeding further we need to verify a technical property:

Proposition 1.2. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

The equivalence class of a path f under the equivalence relation of homotopy will be denoted $[f]$ and called the **homotopy class** of f.

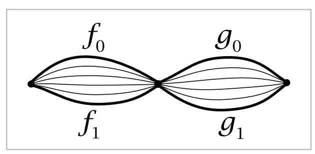
Proof: Reflexivity is evident since $f \simeq f$ by the constant homotopy $f_t = f$. Symmetry is also easy since if $f_0 \simeq f_1$ via f_t, then $f_1 \simeq f_0$ via the inverse homotopy f_{1-t}. For transitivity, if $f_0 \simeq f_1$ via f_t and if $f_1 = g_0$ with $g_0 \simeq g_1$ via g_t, then $f_0 \simeq g_1$ via the homotopy h_t that equals f_{2t} for $0 \le t \le 1/2$ and g_{2t-1} for $1/2 \le t \le 1$. These two definitions agree for $t = 1/2$ since we assume $f_1 = g_0$. Continuity of the associated map $H(s,t) = h_t(s)$ comes from the elementary fact, which will be used frequently without explicit mention, that a function defined on the union of two closed sets is continuous if it is continuous when restricted to each of the closed sets separately. In the case at hand we have $H(s,t) = F(s,2t)$ for $0 \le t \le 1/2$ and $H(s,t) = G(s,2t-1)$ for $1/2 \le t \le 1$ where F and G are the maps $I \times I \to X$ associated to the homotopies f_t and g_t. Since H is continuous on $I \times [0,1/2]$ and on $I \times [1/2,1]$, it is continuous on $I \times I$. □



Given two paths $f,g : I \to X$ such that $f(1) = g(0)$, there is a **composition** or **product path** $f \cdot g$ that traverses first f and then g, defined by the formula

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & 1/2 \le s \le 1 \end{cases}$$

Thus f and g are traversed twice as fast in order for $f \cdot g$ to be traversed in unit time. This product operation respects homotopy classes since if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via homotopies f_t and g_t, and if $f_0(1) = g_0(0)$ so that $f_0 \cdot g_0$ is defined, then $f_t \cdot g_t$ is defined and provides a homotopy $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.



In particular, suppose we restrict attention to paths $f : I \to X$ with the same starting and ending point $f(0) = f(1) = x_0 \in X$. Such paths are called **loops**, and the common starting and ending point x_0 is referred to as the **basepoint**. The set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$.