*§22 The Quotient Topology†

Unlike the topologies we have already considered in this chapter, the quotient topology is not a natural generalization of something you have already studied in analysis. Nevertheless, it is easy enough to motivate. One motivation comes from geometry, where one often has occasion to use "cut-and-paste" techniques to construct such geometric objects as surfaces. The *torus* (surface of a doughnut), for example, can be constructed by taking a rectangle and "pasting" its edges together appropriately, as in Figure 22.1. And the *sphere* (surface of a ball) can be constructed by taking a disc and collapsing its entire boundary to a single point; see Figure 22.2. Formalizing these constructions involves the concept of quotient topology.

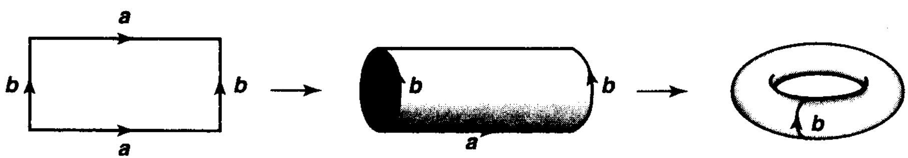

Figure 22.1

†This section will be used throughout Part II of the book. It also is referred to in a number of exercises of Part I.

Definition. Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a ***quotient map*** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition. If X is a space and A is a *set* and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the ***quotient topology*** induced by p.

The topology $\mathcal{T}$ is of course defined by letting it consist of those subsets U of A such that $p^{-1}(U)$ is open in X. It is easy to check that $\mathcal{T}$ is a topology. The sets $\varnothing$ and A are open because $p^{-1}(\varnothing) = \varnothing$ and $p^{-1}(A) = X$. The other two conditions follow from the equations

$$p^{-1}\Big(\bigcup_{\alpha \in J} U_\alpha\Big) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha),$$

$$p^{-1}\Big(\bigcap_{i=1}^{n} U_i\Big) = \bigcap_{i=1}^{n} p^{-1}(U_i).$$

EXAMPLE 3. Let p be the map of the real line $\mathbb{R}$ onto the three-point set $A = \{a, b, c\}$ defined by

$$p(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0. \end{cases}$$

You can check that the quotient topology on A induced by p is the one indicated in Figure 22.3.

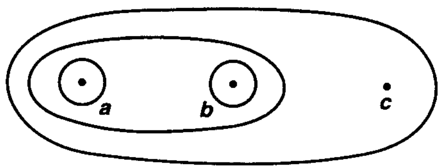

Figure 22.3

There is a special situation in which the quotient topology occurs particularly frequently. It is the following:

Definition. Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a ***quotient space*** of X.

Given X^*, there is an equivalence relation on X of which the elements of X^* are the equivalence classes. One can think of X^* as having been obtained by "identifying" each pair of equivalent points. For this reason, the quotient space X^* is often called an *identification space*, or a *decomposition space*, of the space X.

We can describe the topology of X^* in another way. A subset U of X^* is a collection of equivalence classes, and the set $p^{-1}(U)$ is just the union of the equivalence classes belonging to U. Thus the typical open set of X^* is a collection of equivalence classes whose *union* is an open set of X.

EXAMPLE 4. Let X be the closed unit ball

$$\{x \times y \mid x^2 + y^2 \leq 1\}$$

in $\mathbb{R}^2$, and let X^* be the partition of X consisting of all the one-point sets $\{x \times y\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{x \times y\} \mid x^2 + y^2 = 1\}$. Typical saturated open sets in X are pictured by the shaded regions in Figure 22.4. One can show that X^* is homeomorphic with the subspace of $\mathbb{R}^3$ called the ***unit 2-sphere***, defined by

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

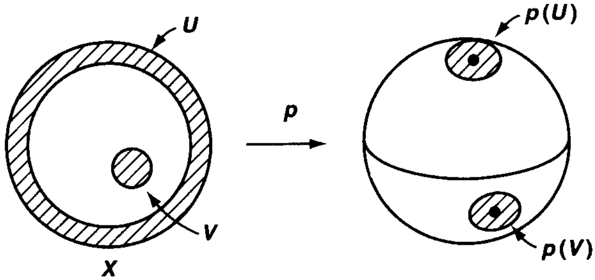

Figure 22.4

EXAMPLE 5. Let X be the rectangle $[0, 1] \times [0, 1]$. Define a partition X^* of X as follows: It consists of all the one-point sets $\{x \times y\}$ where $0 < x < 1$ and $0 < y < 1$, the following types of two-point sets:

$$\{x \times 0, x \times 1\} \quad \text{where } 0 < x < 1,$$
$$\{0 \times y, 1 \times y\} \quad \text{where } 0 < y < 1,$$

and the four-point set

$$\{0 \times 0, 0 \times 1, 1 \times 0, 1 \times 1\}.$$

Typical saturated open sets in X are pictured by the shaded regions in Figure 22.5; each is an open set of X that equals a union of elements of X^*.

The image of each of these sets under p is an open set of X^*, as indicated in Figure 22.6. This description of X^* is just the mathematical way of saying what we expressed in pictures when we pasted the edges of a rectangle together to form a torus.

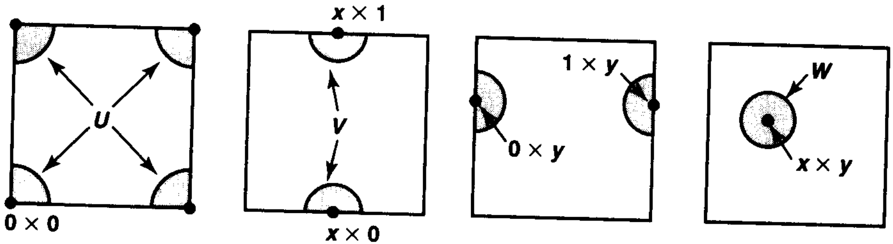

Figure 22.5

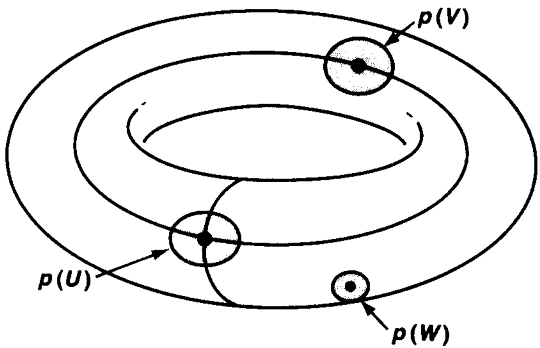

Figure 22.6