7.5 Quotient Spaces

In Chapters 3 and 4 we started with a topological space, for instance a polygonal disk in $\mathbb{R}^2$ or a polyhedral solid in $\mathbb{R}^3$, and glued parts of it together to get a new topological space, for instance T^2 or T^3. What exactly are we doing when we glue? It is clear how we get the points of the new space—they are equivalence classes of points in the old space under the relation that two points are equivalent if they are identical or if we wish to glue them together. However, a topological space is more than a set of points. To make the idea of gluing precise, we need to specify the open sets in the new space. Here is the natural way to do it.

Definition 7.36 *If $\sim$ is an equivalence relation on a topological space X, then the* ***quotient space*** *$X/\sim$ is the space whose points are the equivalence classes of $\sim$. The* ***projection function*** *$\pi : X \to X/\sim$ takes any $x \in X$ to the equivalence class containing x. The* ***quotient topology*** *on $X/\sim$ is defined by*

$$U \text{ is open in } X/\sim \quad \text{if and only if} \quad \pi^{-1}(U) \text{ is open in } X.$$

Example 7.37 Suppose we start with the square disk $\{(x, y) \mid |x| \leq 1, |y| \leq 1\}$ and glue the right edge to the left edge with a half-twist. What do the open sets in the resulting quotient space look like? What is this space?

Solution. The equivalence relation that accomplishes the desired gluing is

$(x_1, y_1) \sim (x_2, y_2)$ if and only if

$$(x_1, y_1) = (x_2, y_2), \text{ or } |x_1| = |x_2| = 1 \text{ and } (x_1, y_1) = (-x_2, -y_2).$$

A subset U of the quotient space is open if and only if its inverse image under the projection function is an open subset of the square. Let's look carefully at what this means. Suppose U contains the equivalence class $P = \{(1, y_0), (-1, -y_0)\}$. Then $\pi^{-1}(U)$ contains both $(1, y_0)$ and $(-1, -y_0)$. Hence for $\pi^{-1}(U)$ to be open, it must contain all points in the square that are within some positive distance ε_1 of $(1, y_0)$, and also all points within some distance ε_2 of $(-1, -y_0)$. See Figure 7.38. Letting ε be the smaller of ε_1 and ε_2, we see that $\pi^{-1}(U)$ must contain all points within ε of either $(1, y_0)$ or $(-1, -y_0)$. When we do the gluing, we can think of these two half-balls as being glued together into one ε-ball about the glued point P. The quotient space is, of course, the Möbius band M^2. ❄

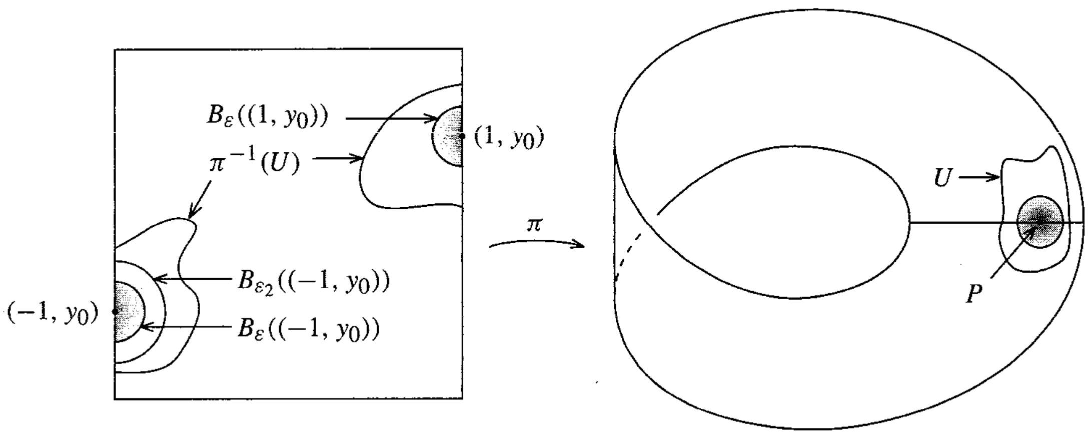

FIGURE 7.38
Gluing the square.

Notice that the definition of the quotient topology makes $\pi : X \to X/\sim$ continuous, and that π is always onto. Hence, by Theorem 7.35, if X is compact, $X/\sim$ will also be compact. Thus, when we take a polygonal region in $\mathbb{R}^2$, which is compact by the Heine-Borel Theorem (Theorem 7.34), and glue its edges together in pairs by identifying points on them, we are guaranteed that the resulting quotient space will be compact. Similarly, if we start with a polyhedral region in $\mathbb{R}^3$ and glue its faces together in pairs, the resulting three-dimensional pseudo-manifold is always compact.