6

The Fundamental Group

Numerical invariants and invariant properties enable us to distinguish certain topological spaces. We can go further and associate with a topological space a set having an algebraic structure. The fundamental group is the most basic of such possibilities. It not only provides a useful invariant for topological spaces, but the algebraic operation of multiplication defined for this group reflects the global structure of the space.

6.1 Deformations with Singularities

A circle, an annulus, a Möbius band, and a solid torus all have the basic shape of a ring going around a hole. Even though these four spaces are topologically distinct, we would like to characterize this ring-like property. We would like to distinguish these spaces from the 2-sphere, for example. The 2-sphere has a hole all right, but the hole of a sphere seems quite a bit different from the hole of a lifesaver. While the 2-dimensional torus has a ring-like shape, it has a hole where the dough of a doughnut would be as well as a hole where you grab to dunk the doughnut into your coffee. We will use loops in the space to get a handle on these kinds of topological properties. Amazing as it may seem, we will be able to detect the ring-like property intrinsically from the space itself, with no need to see how the space is embedded in some ambient Euclidean space.

Recall from Definition 1.40 that a path is a continuous function defined on the closed unit interval $[0, 1]$. The following definition imposes the condition that the function maps the two endpoints to the same point.

Definition 6.1 *A **loop** in a space X is a path $\alpha : [0, 1] \to X$ such that $\alpha(0) = \alpha(1)$. The element $\alpha(0) = \alpha(1)$ of X is the **base point** of the loop α.*

You may recall from Exercise 1 of Section 1.5 the idea of combining two paths into a single path. You simply travel along the first path at twice the normal speed. Then, as long as the final point of the first path coincides with the initial point of the second path, you can continue along the second path at twice the normal speed. For loops at a common base point, we can always perform this kind of splicing.

Definition 6.2 *Suppose the loops α and β in a space X have a common base point x_0. The* **concatenation** *of α with β is the path denoted $\alpha \cdot \beta$ and defined by*

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\ \beta(2s-1) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Since paths α and β with base point x_0 are continuous, we see from the formulas that $\alpha \cdot \beta$ is continuous on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Since

$$\alpha\left(2 \cdot \tfrac{1}{2}\right) = \alpha(1) = x_0 = \beta(0) = \beta\left(2 \cdot \tfrac{1}{2} - 1\right),$$

the two formulas for $\alpha \cdot \beta$ agree at $s = \frac{1}{2}$. By Theorem 1.27, $\alpha \cdot \beta$ is continuous. Of course, $(\alpha \cdot \beta)(0) = \alpha(2 \cdot 0) = \alpha(0) = x_0$ and $(\alpha \cdot \beta)(1) = \beta(2 \cdot 1 - 1) = \beta(1) = x_0$. Hence, the concatenation of two loops based at x_0 is likewise a loop based at x_0.

Figure 6.3 illustrates three typical loops in an annulus. The arrow indicates which way the image is traced as the domain parameter increases from 0 to 1. The loop α does not do any significant traveling around the annulus. In particular, if we lay a string along the image of α, we could wind in the string without letting go of the ends. Even though the loop β ventures forth more boldly, it eventually doubles back. Thus, β can also be deformed back to a constant loop at the base point. Notice that the loop β is not a one-to-one function. Thus, the stages of the deformation will necessarily involve loops that cross themselves. The loop γ goes around the annulus in an essential way. It is clear, at least intuitively, that there is no way to deform γ staying within the annulus, keeping the ends fixed, and finishing with a constant loop.

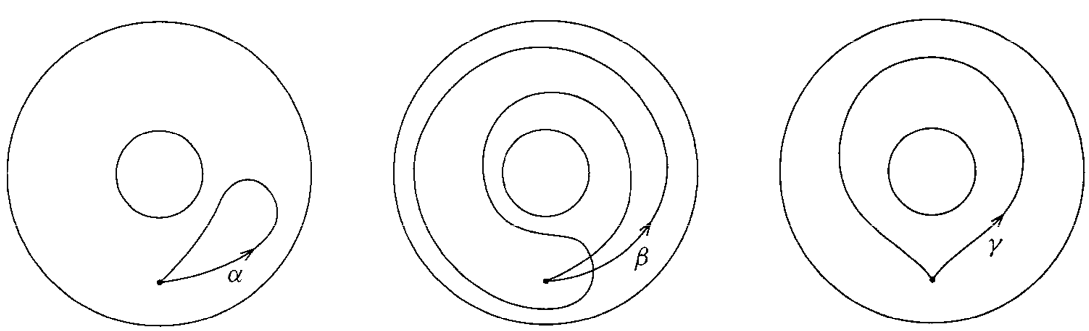

FIGURE 6.3
Typical loops in an annulus.

These deformations of the loops in the annulus illustrate a concept that yields an equivalence relation among loops with a common base point. The following definition makes precise the idea of a continuously parameterized family of loops representing the various stages of a deformation of one loop to another loop. Although continuity is required, the functions involved do not have to be one-to-one. Thus, the loops may cross themselves and they may cross the loops at other stages of the deformation. The loops at all stages of the deformation must map into the space X. Also, the loops at all stages of the deformation must map the end points of the interval to the common base point.

Definition 6.4 *Suppose the loops α and β in a space X have a common base point x_0. A **homotopy** from α to β is a continuous function $H : [0, 1] \times [0, 1] \to X$ such that the loops $H_t : [0, 1] \to X$ defined by $H_t(s) = H(s, t)$ all have the base point x_0 and such that $H_0 = \alpha$ and $H_1 = \beta$. We say the loop α is **homotopic** to the loop β if and only if there is such a homotopy from α to β. This relation is denoted $\alpha \sim \beta$.*

Picture a homotopy from a loop $\alpha : [0, 1] \to X$ to a loop $\beta : [0, 1] \to X$ as mapping a square disk into X. For the ordered pair $(s, t) \in [0, 1] \times [0, 1]$, the t-coordinate is the parameter determining which path $H_t : [0, 1] \to X$ to use and the s-coordinate is the distance parameter along this path. Thus, the bottom of the square maps to X via α, the top of the square maps via β, and the two vertical sides map to the base point x_0.

Figure 6.5 shows how these pieces fit together. The labels in both the domain and range show which pieces correspond to the functions $\alpha = H_0$, $\beta = H_1$, and a typical intermediate loop H_t.

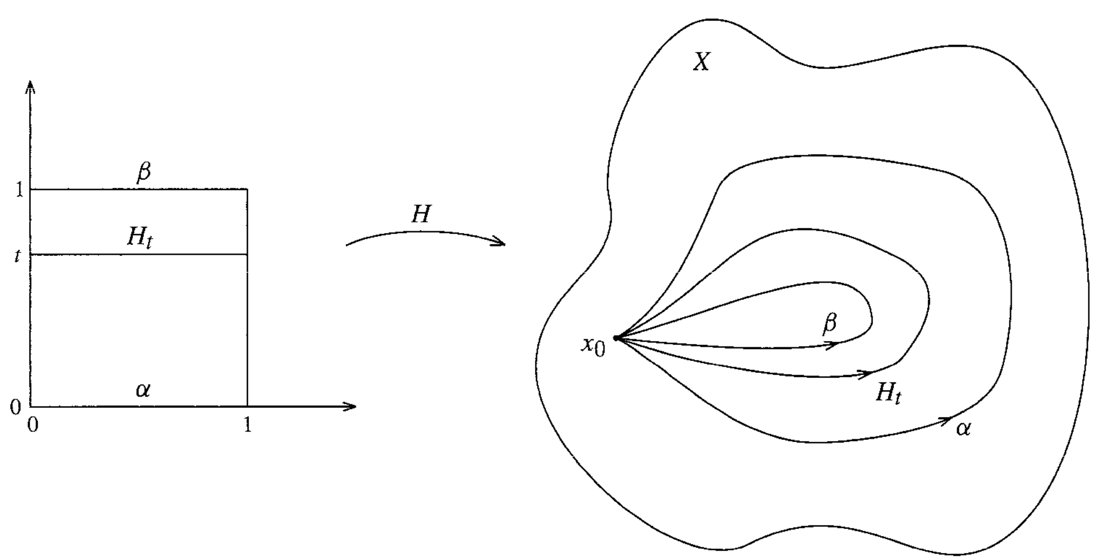

FIGURE 6.5
A homotopy H from a loop α to a loop β.

Example 6.6 Show that any loop in the disk $D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$ is homotopic to the constant loop at the base point.

Solution. Let $\alpha : [0, 1] \to D^2$ denote a loop with base point x_0. For any $s \in [0, 1]$ there is a line segment in D^2 from $\alpha(s)$ to x_0. In fact for $t \in [0, 1]$, $\alpha(s) + t(x_0 - \alpha(s))$ parameterizes this segment as a path. The formula $H(s, t) = \alpha(s) + t(x_0 - \alpha(s))$ defines a continuous function $H : [0, 1] \times [0, 1] \to D^2$. We easily check that H is a homotopy from α to the constant loop at x_0. Indeed,

$$H_0(s) = H(s, 0) = \alpha(s) + 0(x_0 - \alpha(s)) = \alpha(s),$$
$$H_1(s) = H(s, 1) = \alpha(s) + 1(x_0 - \alpha(s)) = x_0,$$

and for all $t \in [0, 1]$,

$$H_t(0) = H(0, t) = \alpha(0) + t(x_0 - \alpha(0)) = x_0 + t(x_0 - x_0) = x_0,$$
$$H_t(1) = H(1, t) = \alpha(1) + t(x_0 - \alpha(1)) = x_0 + t(x_0 - x_0) = x_0.$$

❄

Theorem 6.7 *The relation of homotopy among loops with base point x_0 in a space X is an equivalence relation on the set of all such loops.*

Proof. We simply cook up homotopies to verify the reflexive, symmetric, and transitive properties as required by Definition 1.2. Exercise 7 at the end of this section asks you to check that the given formulas do indeed define the required homotopies. ❄

We are more interested in the net amount of winding a loop does in a space than in the details of how fast it travels or any backtracking it does. Thus, we will be more interested in the equivalence class of a loop than in the loop itself. Recall from Exercise 2 of Section 1.1 that the set of all loops in a space at a given base point is the disjoint union of these equivalence classes.

Definition 6.8 *For a loop α with base point x_0 in a space X, the set of all loops homotopic to α is the* **homotopy class** *of α. This set is denoted $\langle\alpha\rangle$.*

In the next section, we will see how concatenation of loops leads to a kind of multiplication on the homotopy classes of loops. The resulting set with its algebraic operation will be a topological invariant of the underlying space.

6.2 Algebraic Properties

In the previous section we saw how to concatenate two loops to produce a new loop. The goal of this section is to use concatenation to define an algebraic structure that will give us some information about the shape of a topological space. We run into several immediate difficulties if we try to work with the loops themselves. For example, we would like a constant path ε to act as the algebraic identity element. However, for any reasonably interesting path $\alpha : [0, 1] \to X$, the concatenation $\alpha \cdot \varepsilon$ will not equal α. Similarly, we would like to be able to cancel a loop by concatenating it with a loop that traces the path of the original loop in the reverse direction. But again, for a nonconstant loop, there is no way to concatenate it with any loop to produce a constant loop.

The solution to this problem is to consider homotopy classes of loops rather than the loops themselves. This also allows us to concentrate on the essential shape of the space without being distracted by the meandering of a loop through the space. Here then is the definition of the product of two homotopy classes of loops.

Definition 6.9 *Let $\alpha : [0, 1] \to X$ and $\beta : [0, 1] \to X$ be two loops with base point x_0 in a space X. The **product** of the homotopy classes $\langle\alpha\rangle$ and $\langle\beta\rangle$ of these two loops is denoted $\langle\alpha\rangle\langle\beta\rangle$ and is defined to be the homotopy class $\langle\alpha \cdot \beta\rangle$ of the concatenation of α and β.*

The above definition uses two loops α and β to represent their homotopy classes. This raises the question as to whether the product of the classes depends on the choice of the representatives. That is, if $\alpha' \sim \alpha$ and $\beta' \sim \beta$, then $\langle\alpha\rangle = \langle\alpha'\rangle$ and $\langle\beta\rangle = \langle\beta'\rangle$. We need to verify in this situation that $\langle\alpha\rangle\langle\beta\rangle = \langle\alpha'\rangle\langle\beta'\rangle$. This is a typical example of showing that an operation is well-defined.

Theorem 6.10 *Suppose $\alpha \sim \alpha'$ and $\beta \sim \beta'$ as loops based at x_0 in a space X. Then $\langle\alpha\rangle\langle\beta\rangle = \langle\alpha'\rangle\langle\beta'\rangle$.*

Proof. Since $\langle\alpha\rangle\langle\beta\rangle = \langle\alpha \cdot \beta\rangle$ and $\langle\alpha'\rangle\langle\beta'\rangle = \langle\alpha' \cdot \beta'\rangle$, we need to show that $\alpha \cdot \beta \sim \alpha' \cdot \beta'$. We use a homotopy F from α to α' and a homotopy G from β to β' to paste together a homotopy H from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$. Figure 6.11 provides a guide for pasting together these homotopies. The domain $[0, 1] \times [0, 1]$ is labeled with the functions used on the various pieces.

Define H by

$$H(s,t) = \begin{cases} F(2s, t) & \text{for } 0 \le s \le \frac{1}{2}, \\ G(2s - 1, t) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$

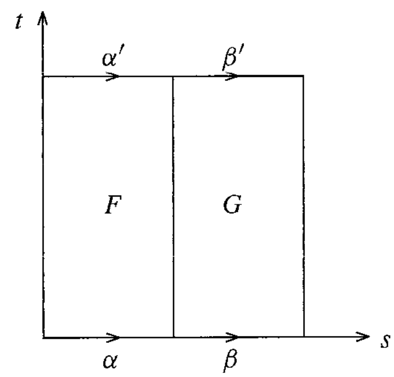

FIGURE 6.11
A guide to defining a homotopy from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$.

Notice that domains of the two portions of the definition overlap along the line where $s = \frac{1}{2}$. For these points we have $F(2s, t) = F(1, t) = x_0 = G(0, t) = G(2s - 1, t)$. Therefore H is well-defined, and by Theorem 1.27 it is continuous. Now at any stage $t \in [0, 1]$, we have that $H_t = F_t \cdot G_t$. Therefore, H_t is a loop with base point x_0. It follows that H is a homotopy from $H_0 = F_0 \cdot G_0 = \alpha \cdot \beta$ to $H_1 = F_1 \cdot G_1 = \alpha' \cdot \beta'$. ❈

Having verified that concatenation of loops gives a well-defined product of homotopy classes of loops, we are now ready to consider the algebraic properties of this multiplication.

Theorem 6.12 *Let x_0 be a common base point for all loops in a space X. The product of homotopy classes of loops in X satisfies the following three properties:*

Associativity: *For any loops α, β, and γ in X, we have $(\langle\alpha\rangle\langle\beta\rangle)\langle\gamma\rangle = \langle\alpha\rangle(\langle\beta\rangle\langle\gamma\rangle)$.*

Existence of an identity element: *The constant path ε defined by $\varepsilon(s) = x_0$ for all $s \in [0, 1]$ determines a homotopy class $\langle\varepsilon\rangle$ that satisfies $\langle\alpha\rangle\langle\varepsilon\rangle = \langle\alpha\rangle = \langle\varepsilon\rangle\langle\alpha\rangle$ for any loop α.*

Existence of inverses: *For any loop α, the reverse loop α^{-1} defined by $\alpha^{-1}(s) = \alpha(1 - s)$ determines a homotopy class $\langle\alpha^{-1}\rangle$ that satisfies $\langle\alpha\rangle\langle\alpha^{-1}\rangle = \langle\varepsilon\rangle$ and $\langle\alpha^{-1}\rangle\langle\alpha\rangle = \langle\varepsilon\rangle$.*

Proof. The proofs of these three properties involve deforming the parameterization of the loops involved to form the required homotopies. The ideas are quite simple, although the details of writing down the formulas is more of a technical exercise in analytic geometry.

Associativity: From the definition of multiplication in terms of concatenation, $(\langle\alpha\rangle\langle\beta\rangle)\langle\gamma\rangle = \langle\alpha \cdot \beta\rangle\langle\gamma\rangle = \langle(\alpha \cdot \beta) \cdot \gamma\rangle$ and $\langle\alpha\rangle(\langle\beta\rangle\langle\gamma\rangle) = \langle\alpha\rangle\langle\beta \cdot \gamma\rangle = \langle\alpha \cdot (\beta \cdot \gamma)\rangle$. Thus, we need a homotopy from the loop $(\alpha \cdot \beta) \cdot \gamma$ to the loop $\alpha \cdot (\beta \cdot \gamma)$. Figure 6.13 provides a guide to defining the required homotopy

$$F(s,t) = \begin{cases} \alpha\left(\dfrac{4s}{1+t}\right) & \text{for } 0 \le s \le \dfrac{1+t}{4}, \\ \beta(4s-1-t) & \text{for } \dfrac{1+t}{4} \le s \le \dfrac{2+t}{4}, \\ \gamma\left(\dfrac{4s-2-t}{2-t}\right) & \text{for } \dfrac{2+t}{4} \le s \le 1. \end{cases}$$

Exercise 2 at the end of this section asks you to verify that F does the job required of a homotopy from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$.

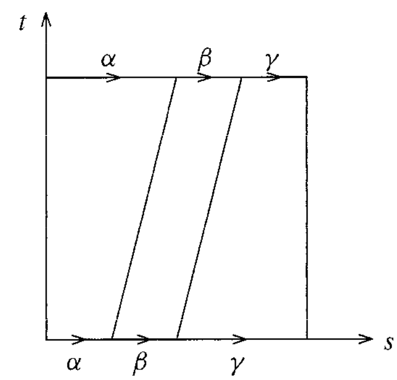

FIGURE 6.13
A guide to defining a homotopy from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$.

Existence of an identity element: Since $\langle\varepsilon\rangle\langle\alpha\rangle = \langle\varepsilon \cdot \alpha\rangle$, we need to find a homotopy from the loop $\varepsilon \cdot \alpha$ to the loop α. Exercise 3 at the end of this section asks you to draw the guide to construction of such a homotopy G and to verify the formulas in the following definition:

$$G(s,t) = \begin{cases} x_0 & \text{for } 0 \le s \le \dfrac{1}{2} - \dfrac{1}{2}t, \\ \alpha\left(\dfrac{2s-1+t}{1+t}\right) & \text{for } \dfrac{1}{2} - \dfrac{1}{2}t \le s \le 1. \end{cases}$$

Exercise 4 at the end of this section asks you to construct a similar homotopy from $\alpha \cdot \varepsilon$ to α to show that $\langle\alpha\rangle\langle\varepsilon\rangle = \langle\alpha\rangle$.

Existence of inverses: Since $\langle\alpha\rangle\langle\alpha^{-1}\rangle = \langle\alpha \cdot \alpha^{-1}\rangle$ we need to find a homotopy from the loop $\alpha \cdot \alpha^{-1}$ to the constant loop ε. The easiest way to construct the desired homotopy is to have $H_t(s)$ travel along α as far as $\alpha(1-t)$ and stay there until there is just enough time left to return to the base point along α^{-1}. The following definition gives the desired homotopy:

$$H(s,t) = \begin{cases} \alpha(2s) & \text{for } 0 \le s \le \dfrac{1-t}{2}, \\ \alpha(1-t) & \text{for } \dfrac{1-t}{2} \le s \le \dfrac{1+t}{2}, \\ \alpha(2-2s) & \text{for } \dfrac{1+t}{2} \le s \le 1. \end{cases}$$

Since the reverse of α^{-1} is α, we can interchange the roles of α and α^{-1} in the definition of this homotopy to obtain a homotopy from $\alpha^{-1} \cdot \alpha$ to ε. This shows that $\langle \alpha^{-1} \rangle \langle \alpha \rangle = \langle \varepsilon \rangle$.

❄

Associativity, the existence of an identity, and the existence of inverses are the three properties that define the algebraic structure known as a **group**. Thus, the previous theorem can be summarized as stating that the collection of homotopy classes of loops forms a group under the multiplication. Theorem 6.12 justifies the terminology given in the following definition.

Definition 6.14 *Let X be a topological space with base point x_0. The set of homotopy classes of loops in X based at x_0 along with the operation of forming products of homotopy classes is the* ***fundamental group*** *of the space X based at x_0. This group is denoted $\pi_1(X, x_0)$.*

We have defined the fundamental group in terms of homotopy classes of functions defined on an interval. Topologists have considered the analogous situation with functions defined on the n-fold Cartesian product of intervals. This leads to the higher homotopy groups $\pi_n(X, x_0)$ of which the fundamental group is the first. Many of our results about the fundamental group extend to the higher homotopy groups.

6.3 Invariance of the Fundamental Group

With any topological space and designated base point in the space, we have associated an algebraic structure known as the fundamental group. We want to obtain some geometric information about the space from the fundamental group. In particular, we want to use the fundamental group as a topological invariant of the space. In this section we will examine the mathematical concepts that make the fundamental group a valuable tool in studying a topological space.

The first question is the algebraic version of the problem of identity: what does it mean for two groups to be the same? As with geometric figures, we will give different answers to the question depending on the information we are after. We typically do not want to insist that the elements are literally identical. No two distinct objects would be identified under this relation. However, we usually want more than just a bijection between the set of elements that comprise the group. This would tell us nothing about the algebraic structure of the underlying sets. The most fruitful concept of two groups being essentially the same is captured in the following definition.

Definition 6.15 *Let G and H be groups. Denote the group operation in both by juxtaposition of the elements. A* ***homomorphism*** *from G to H is a function $h : G \to H$ such that $h(ab) = h(a)h(b)$ for any two elements $a, b \in G$. An* ***isomorphism*** *is a homomorphism that is a bijection. A group G is* ***isomorphic*** *to a group H if and only if there is an isomorphism from G to H. We write $G \cong H$ to denote this relation.*

Thus, an isomorphism respects the set theoretical structure of the groups as well as the algebraic structure. When we try to determine the fundamental group of a space, we will be satisfied to know it is isomorphic to some group that we can describe in terms of familiar mathematical objects (the integers, reflections and rotations of polygons, strings of characters with certain cancellation rules, and so forth).

Recall that a path component of a space is the set of all points that can be joined to a given point by a path. Because loops are paths, the loops in a space all lie in the path component containing the base point. Thus, it is reasonable to consider the fundamental group only for spaces that are path-connected. Once we adopt this convention, the following theorem says that we get the same group (up to isomorphism) no matter what base point we designate in a space.

Theorem 6.16 *Suppose X is a path-connected space. For any two points x_0 and x_1 in X, the group $\pi_1(X, x_0)$ is isomorphic to the group $\pi_1(X, x_1)$.*

Proof. The proof of this theorem consists of defining an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ and verifying that the function satisfies the various properties required to confirm that it is indeed an isomorphism. The significant steps in this proof are outlined below. The details are left to you to supply as the solution to Exercise 6 at the end of this section. This is an excellent opportunity for you to demonstrate your ability to construct homotopies.

Let γ be a path from x_0 to x_1. A loop based at x_0 can have its base point switched to x_1 by traveling along γ in the reverse direction (from x_1 to x_0), then traversing the loop, and returning to x_1 by traveling along γ in the forward direction. More formally, we define a function $\Gamma : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\Gamma(\langle\alpha\rangle) = \langle\gamma^{-1} \cdot \alpha \cdot \gamma\rangle$ for any loop α based at x_0. There are several details for you to check about this definition of Γ. These are spelled out in parts (a) through (e) of Exercise 6.

For any loops α, β based at x_0, part (g) of Exercise 6 gives an easy way to prove that $\Gamma(\langle\alpha\rangle\langle\beta\rangle) = \Gamma(\langle\alpha\rangle)\Gamma(\langle\beta\rangle)$. That is, Γ is a homomorphism.

The way we used the path γ to construct Γ can be applied using the reverse path γ^{-1} to construct a function $\Gamma' : \pi_1(X, x_1) \to \pi_1(X, x_0)$. As you can check in part (h) of Exercise 6, this function is the inverse of Γ. Hence Γ is a bijection. ❄

The next theorem is the key to proving that the fundamental group is a topological invariant. For a continuous function between topological spaces, we can compose this function with any loop in the domain to get a loop in the range. When we view this as a transformation of homotopy classes of loops, this correspondence is a homomorphism between the fundamental groups of the spaces.

Theorem 6.17 *Suppose $f : X \to Y$ is a continuous function and x_0 is designated as the base point in X. Then f induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$ for all $\langle\alpha\rangle \in \pi_1(X, x_0)$.*

Proof. You will find the proof of this theorem is more interesting to work out on your own than it would be to read. There are three main steps: check that $f \circ \alpha$ is indeed a loop in Y based at $f(x_0)$, show that f_* is well-defined, and confirm that f_* is a homomorphism. Exercise 7 at the end of this section asks you to fill in the details. ❄

Now all the pieces are in place to show that the fundamental group is a topological invariant. This is most easily done as a consequence of the properties stated in the following theorem.

Theorem 6.18 *Suppose X, Y, and Z are topological spaces. Let x_0 be designated as the base point for X.*

1. *The identity function $\mathrm{id}_X : X \to X$ induces the identity homomorphism $\mathrm{id}_{\pi_1(X,x_0)} : \pi_1(X, x_0) \to \pi_1(X, x_0)$.*
2. *If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $(f \circ g)_* = f_* \circ g_*$.*

Proof. For any element $\langle\alpha\rangle \in \pi_1(X, x_0)$, we have

$$\begin{aligned}(\mathrm{id}_X)_*(\langle\alpha\rangle) &= \langle \mathrm{id}_X \circ \alpha\rangle \\ &= \langle\alpha\rangle \\ &= \mathrm{id}_{\pi_1(X,x_0)}(\langle\alpha\rangle).\end{aligned}$$

Hence, $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$.

We also have

$$\begin{aligned}(f \circ g)_*(\langle\alpha\rangle) &= \langle (f \circ g) \circ \alpha\rangle \\ &= \langle f \circ (g \circ \alpha)\rangle \\ &= f_*(\langle g \circ \alpha\rangle) \\ &= f_*(g_*(\langle\alpha\rangle)) \\ &= (f_* \circ g_*)(\langle\alpha\rangle).\end{aligned}$$

Hence, $(f \circ g)_* = f_* \circ g_*$. ❄

In the language of category theory, the previous theorem says that the assignment of the fundamental group to a topological space and the assignment of the induced homomorphism to a continuous function is a **covariant functor**. Watch how easily the topological invariance follows from the two conditions in Theorem 6.18.

Theorem 6.19 *The fundamental group is a topological invariant for path-connected topological spaces.*

Proof. Suppose $h : X \to Y$ is a homeomorphism between path-connected spaces X and Y. Let x_0 be designated as the base point of X. Because h is a homeomorphism, h^{-1} exists, and both h and h^{-1} are continuous. Therefore, $(h^{-1})_* \circ h_* = (h^{-1} \circ h)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$, and $h_* \circ (h^{-1})_* = (h \circ h^{-1})_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y,f(x_0))}$. It follows that the homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$ induced by the homeomorphism $h : X \to Y$ has an inverse $(h^{-1})_*$. Thus, h_* is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, h(x_0))$.

By Theorem 6.16, any choice of base points for X and Y will give groups isomorphic to $\pi_1(X, x_0)$ and $\pi_1(Y, h(x_0))$. Therefore, the fundamental group depends only on the topological type of the space and not on the choice of base point. ❄