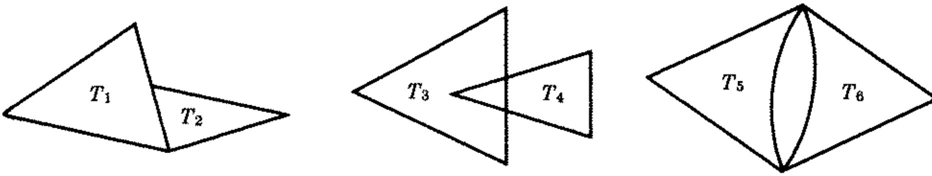

FIGURE 1.12 Some types of intersection forbidden in a triangulation.

Definition A *triangulation* of a compact surface S consists of a finite family of closed subsets $\{T_1, T_2, \ldots, T_n\}$ that cover S, and a family of homeomorphisms $\varphi_i : T'_i \to T_i$, $i = 1, \ldots, n$, where each T'_i is a triangle in the plane $\mathbf{R}^2$ (i.e., a compact subset of $\mathbf{R}^2$ bounded by three distinct straight lines). The subsets T_i are called "triangles." The subsets of T_i that are the images of the vertices and edges of the triangle T'_i under φ_i are also called "vertices" and "edges," respectively. Finally, it is required that any two distinct triangles, T_i and T_j, either be disjoint, have a single vertex in common, or have one entire edge in common.

Perhaps the conditions in the definition are clarified by Figure 1.12, which shows three *unallowable* types of intersection of triangles.

Given any compact surface S, it seems plausible that there should exist a triangulation of S. A rigorous proof of this fact (first given by T. Radó in 1925) requires the use of a strong form of the Jordan curve theorem. Although it is not difficult, the proof is tedious, and we will not repeat it here.

We can regard a triangulated surface as having been constructed by gluing together the various triangles in a certain way, much as we put together a jigsaw puzzle or build a wall of bricks. Because two different triangles cannot have the same vertices we can specify completely a triangulation of a surface by numbering the vertices, and then listing which triples of vertices are vertices of a triangle. Such a list of triangles completely determines the surface together with the given triangulation up to homeomorphism.

Examples

6.1 The surface of an ordinary tetrahedron in Euclidean 3-space is homeomorphic to the sphere S^2; moreover, the four triangles satisfy all the conditions for a triangulation of S^2. In this case there are four vertices, and every triple of vertices is the set of vertices of a triangle. No other triangulation of any surface can have this property.

6.2 In Figure 1.13 we show a triangulation of the projective plane, considered as the space obtained by identifying diametrically opposite points on the bound-

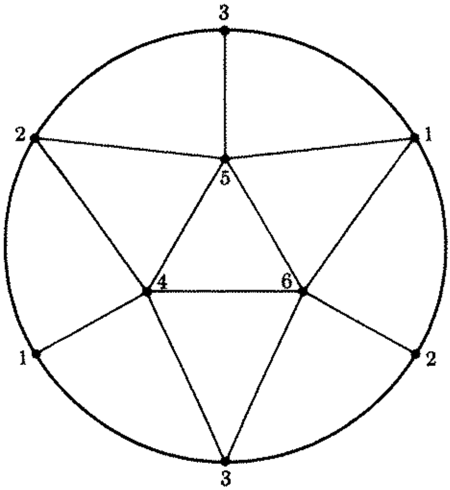

FIGURE 1.13 A triangulation of the projective plane.

ary of a disc. The vertices are numbered from 1 to 6, and there are the following 10 triangles:

124	245
235	135
156	126
236	346
134	456

6.3 In Figure 1.14 we show a triangulation of a torus, regarded as a square

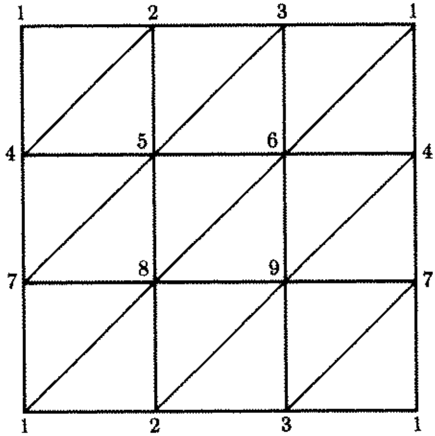

FIGURE 1.14 A triangulation of a torus.