Connectedness

Definition 0.0.1: (X, τ) is **disconnected** if there exist nonempty open sets U, V such that $U \cap V = \emptyset$ and $X = U \cup V$.

Also, a space is **connected** if it is not disconnected.

Exercise 0.0.2: Show that (X, τ) is disconnected if and only if
a.) there exist nonempty *closed* sets U, V such that $U \cap V = \emptyset$ and $X = U \cup V$;
b.) there exists a proper subset $U \subset X$ which is both open and closed in X.

Exercise 0.0.3: Think through: any X with the anti-discrete topology is connected. Any X (that has at least two points) with the discrete topology is disconnected.

Definition 0.0.4: Given (X, τ), a subset $A \subset X$ is **disconnected** if the topological space (A, τ_A) is disconnected, where τ_A is the subspace topology on A.

Also, a A is **connected** if it is not disconnected.

Example 0.0.5: a.) Let $X = [0,1] \cup [2,3]$ (with the subspace topology of the usual topology of $\mathbb{R}$). Then $U = [0,1]$ and $V = [2,3]$ are open in X, disjoint, whose union is X, so X is disconnected.

b.) Let $X = \mathbb{Q}$ (again with the subspace topology of the usual topology of $\mathbb{R}$). Then clearly,

$$X = [(-\infty, \sqrt{2}) \cap \mathbb{Q}] \,\cup\, [(\sqrt{2}, \infty) \cap \mathbb{Q}]$$

where $U = (-\infty, \sqrt{2}) \cap \mathbb{Q}$ and $V = (\sqrt{2}, \infty) \cap \mathbb{Q}$ are open and disjoint in $\mathbb{Q}$, so $\mathbb{Q}$ is disconnected.

Note that $U = (-\infty, \sqrt{2}) \cap \mathbb{Q} = (-\infty, \sqrt{2}] \cap \mathbb{Q}$ so that U is also closed in $\mathbb{Q}$ (and similarly V is closed in $\mathbb{Q}$ as well.)

To get many examples of connected sets, next we will show that every interval I is connected in $\mathbb{R}$ (if $\mathbb{R}$ has the usual topology). So here I is one of
(a, b) where $\infty \leq a < b \leq \infty$ or
$(a, b]$ where $\infty \leq a < b < \infty$ or
$[a, b)$ where $\infty < a < b \leq \infty$ or
$[a, b]$ where $\infty < a < b < \infty$ where, by definition
$I = (a, b) = \{z \in \mathbb{R} \,|\, a < z < b\}$ with the other cases defined similarly.

Lemma 0.0.6: Every interval I of $\mathbb{R}$ is connected.

Proof: This proof is for $I = (-\infty, \infty)$. The other cases can be proved similarly.

Assume, by contradiction, that $I = \mathbb{R} = (-\infty, \infty)$ is not connected (i.e. disconnected). We will use that this means there exist non-empty, disjoint, *closed* sets U, V in $\mathbb{R}$ such that $\mathbb{R} = U \cup V$.

Since U, V are non-empty, there exist $a_0 \in U$ and $b_0 \in V$. Consider $z = \frac{a_0 + b_0}{2}$. Clearly, $a_0 < z < b_0$, so $z \in I$ and thus either $z \in U$ or $z \in V$.

Case 1: If $z \in U$, then let $a_1 = z$ and $b_1 = b_0$.

Case 2: If $z \in V$, then let $a_1 = a_0$ and $b_1 = z$.

Note that with this notation we have $a_0 \leq a_1 < b_1 \leq b_0$.

By induction, for each $n \in \mathbb{N}$, given $a_n < b_n$ such that $a_n \in U$ and $b_n \in V$, consider $z = \frac{a_n+b_n}{2}$. Clearly, $a_n < z < b_n$, so $z \in I$ and thus either $z \in U$ or $z \in V$.

Case 1: If $z \in U$, then let $a_{n+1} = z$ and $b_{n+1} = b_n$.

Case 2: If $z \in V$, then let $a_{n+1} = a_n$ and $b_{n+1} = z$.

Note that with this notation we have two sequences

$$a_0 \leq a_1 \leq ... \leq a_n \leq ...$$

and

$$b_0 \geq b_1 \geq b_2 \geq ... \geq b_n \geq ...$$

where $a_i \in U$ and $b_i \in V$ $\forall i \in \mathbb{N}$ and also $a_i < b_j$ $\forall i, j \in \mathbb{N}$.

Since (a_i) is monotone increasing and bounded from above by e.g. b_1, it is convergent, moreover $\lim_{\rightarrow i \rightarrow \infty} a_i = \sup_i\{a_i\}$. Denote this number by L.

Also, since (b_i) is monotone decreasing and bounded from below by e.g. a_1, it is convergent, moreover $\lim_{\rightarrow i \rightarrow \infty} b_i = \inf_i\{b_i\}$. Denote this number by M.

In addition, $b_i - a_i = \frac{b_0-a_0}{2^i}$ so that $\lim_{\rightarrow i \rightarrow \infty} (b_i - a_i) = 0$. Since the $(a_i), (b_i)$ sequences are convergent, we have $0 = \lim_{\rightarrow i \rightarrow \infty} (b_i - a_i) = \lim_{\rightarrow i \rightarrow \infty} b_i - \lim_{\rightarrow i \rightarrow \infty} a_i = M - L$ so $L = M$.

But $a_i \in U \forall i$ and U is closed in $I = \mathbb{R}$, so $L = M \in U$. Similarly, $b_i \in V \forall i$ and V is closed in $I = \mathbb{R}$ $M = L \in V$. But then $U \cap V \neq \emptyset$ and that is a contradiction. ■

Connectedness is a topological invariant, so it helps distinguish topological spaces. That it is a topological invariant follows from the next lemma.

Lemma 0.0.7: If $f : X \rightarrow Y$ is continuous and onto and X is connected, then Y is connected as well.

Proof: Assume by contradiction that Y is disconnected, that is, there exist non-empty, disjoint open sets U, V such that $Y = U \cup V$.

Since f is continuous, we have $W = f^{-1}(U) \in \tau_X$ and $Z = f^{-1}(V) \in \tau_X$.

The assumptions on U, V imply that W, Z are non-empty and disjoint with $X = W \cup Z$, so X is disconnected. But that is a contradiction. ■

Example 0.0.8: Classify the intervals $[0, 1]$, $[0, 1)$ and $(0, 1)$ up to homeomorphism. That is, decide which are homeomorphic and which are not.

Solution: Let $A = [0, 1]$, $B = [0, 1)$ and $C = (0, 1)$.

$A = [0, 1]$ is compact, since it is a closed and bounded set in $\mathbb{R}$. (Since we are in $\mathbb{R}$ the Heine-Borel theorem applies.)

The sets B, C are not closed, so they are not compact. Compactness is a topological invariant, so A is not homeomorphic to either B or C.

Now, we claim that B is not homeomorphic to C either. Since, suppose $\exists f : B \rightarrow C$ homeomorphism. Then the restriction

$$f|_{(0,1)} : (0, 1) \rightarrow (0, 1) \setminus \{f(0)\}$$

would be a homeomorphism too. (We showed in a homework that the restriction of a homeomorphism is a homeomorphism.)

However, $(0,1)$ is connected, while $(0,1) \setminus \{f(0)\}$ is not connected. This is because

$$(0,1) \setminus \{f(0)\} = (0, f(0)) \cup (f(0), 1)$$

provides $(0,1)$ as a disjoint union of its two non-empty open subsets $U = (0, f(0))$ and $V = (f(0), 1)$. (The point is that these U and V are open in $X = (0,1)$.)

Path-connectedness

Definition 0.0.9: A path is a continuous mapping $\gamma : [0,1] \to X$.

We say that the path γ begins at $\gamma(0) = a$ and ends at $\gamma(1) = b$.

Note that, since $[0,1]$ is compact and connected in $\mathbb{R}$ and γ is continuous, we have that the image $Im(\gamma) \subset X$ is also compact and connected.

Definition 0.0.10: (X, τ) is **path connected** if $\forall\, a, b \in X$, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Theorem 0.0.11: If a topological space (X, τ_X) is path connected, then it is connected.

Proof: Suppose the space is not connected. Then there exist disjoint, non-empty open sets $U, V \subset X$ such that $X = U \cup V$.

Pick $a \in U$ and $b \in V$ (which exist, since U, V are not empty). Since X is path connected $\exists \gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Clearly, for the image of gamma we have

$$Im(\gamma) = [Im(\gamma) \cap U] \cup [Im(\gamma) \cap V]$$

where $[Im(\gamma) \cap U]$ and $[Im(\gamma) \cap V]$ are non-empty, disjoint, open subsets of $Im(\gamma)$.

Thus $Im(\gamma)$ is disconnected. But it is the image of a connected set under a continuous map, so must be connected. ■

However, the reverse is not true: connectedness does not imply path-connectedness. Here is a classical example of a topological space that is connected, but not path-connected.

Example 0.0.12 (The Topologist's Sine Curve): Let $Y = \{(x, \sin(1/x)) \;:\; x > 0\} \subset \mathbb{R}^2$, that is: Y is (part of) the graph of $y = \sin(\frac{1}{x})$ in $\mathbb{R}^2$. Note that Y is connected, since it is path-connected.

The topologist's sine curve is $\mathrm{cl}(Y) = \{(0, b) \;:\; -1 \le b \le 1\} \cup Y$. We prove the following lemma to show that $\mathrm{cl}(Y)$ is connected

Lemma 0.0.13: If $A \subset X$ is connected, then $\mathrm{cl}(A)$ is also connected.

Caution: be careful which topology you are considering. Here, when we say that A is connected, we say that it cannot be written as the union of two disjoint, non-empty sets open in A. But when we prove $\mathrm{cl}(A)$ is connected, we have to show that there are no disjoint non-empty subsets open in $\mathrm{cl}(A)$ whose union is $\mathrm{cl}(A)$.

Proof:

Suppose, by way of contradiction, that $\text{cl}(A) = U \cup V$ for some nonempty, disjoint U, V that are open in $\text{cl}(A)$.

Since $U = cl(A) \setminus V$ and $V = cl(A) \setminus U$, we also have that U, V are closed in $\text{cl}(A)$.

We showed before that this means: there exist W, Z closed in X such that $U = W \cap \text{cl}(A)$ and $V = Z \cap \text{cl}(A)$.

Then

$$A = A \cap \text{cl}(A) = A \cap [U \cup V] = [A \cap U] \cup [A \cap V]$$

Also, $A \cap U = A \cap W \cap cl(A) = A \cap W$ and $A \cap V = A \cap Z \cap cl(A) = A \cap Z$ so

$$A = [A \cap W] \cup [A \cap Z] \quad (*)$$

where $A \cap W$ and $A \cap Z$ are closed in A, since W, Z are closed in X. Also, this is a disjoint union, since U and V are disjoint.

However, A is connected so (*) implies that one of $A \cap W$ or $A \cap Z$ must be empty. Without loss of generality, assume that $A \cap Z = \emptyset$. So $A = A \cap W$ and therefore $A \subset W$. But then $\text{cl}(A) \subset W$, since W is closed in X.

Thus, $U = W \cap \text{cl}(A) = cl(A)$, so $V = \emptyset$ and that contradicts our initial assumption.

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Back to the Topologist's Sine curve: by the lemma we just proved, since Y is connected, we have $\text{cl}(Y) = Y \cup \{(0, b) \ : \ -1 \leq b \leq 1\}$ is connected too.

However, $\text{cl}(Y)$ is not path connected as a point in $\text{cl}(Y) \setminus Y$ cannot be connected to a point of Y by a path.

Here is an outline of the proof - we will show that there is no path beginning at the origin and ending at a point of Y. The proof is by contradiction.

Let $(x_1, \sin \frac{1}{x_1}) \in Y$ and assume that there is a $\gamma : [0, 1] \to \text{cl}(Y)$ path (i.e. continuous function) such that $\gamma(0) = (0, 0)$ and $\gamma(1) = (x_1, \sin \frac{1}{x_1})$.

First we will show that there is a point in $[0, 1]$ when γ "leaves the y-axis".

Consider the set $W := \{t \in [0, 1] \,|\, \gamma(t) \in \{(0, b) \ : \ -1 \leq b \leq 1\}\}$. W is not empty, since 0 is in it. Let $t' = \sup\{t \in W\}$. By the continuity of γ, $t' \in W$, since if $t_n \to t'$, $t_n \in W$, $\forall n \in \mathbb{N}$, we must have $\gamma(t_n) \to \gamma(t')$ (Sequences $t_n \to t'$, $t_n \in W$, $\forall n \in \mathbb{N}$ exist, by definition of suprema.)

Now, consider pr_x, the projection on x, which is also continuous. We then have $(pr_x \circ \gamma)(t_n) \to (pr_x \circ \gamma)(t')$, by continuity of $pr_x \circ \gamma$. But $(pr_x \circ \gamma)(t_n) = 0$, so $(pr_x \circ \gamma)(t') = 0$, which means $\gamma(t') = (0, b_0)$ for some $b_0 \in [-1, 1]$.

Thus we know $\gamma(t) \in Y \ \forall t > t'$.

Assume first that $t' = 0$, so $b_0 = 0$, since $\gamma(0) = (0, 0)$. We then have $\gamma(t) \in Y \ \forall t \in (0, 1]$.

We will find a sequence $(t_n) \subset (0, 1]$ such that $t_n \to 0$, but $\gamma(t_n)$ is an alternating subsequence of $(x_m, (-1)^m)$, $x_m \to 0$, $m \in \mathbb{N}$ so that $\gamma(t_n) \not\to \gamma(0) = (0, 0)$ which contradicts γ being continuous.

We will use the fact that $\sin(\frac{\pi}{2} + m\pi) = (-1)^m$ for $m \in N$ to construct this sequence.

We will also make use of the Intermediate Value theorem according to which if $g : [a, b] \to \mathbb{R}$ is continuous then if (wlog) $g(a) < g(b)$ we have for all $c \in (g(a), g(b))$ there is a $z \in (a, b)$

for which $g(z) = c$. (This is a consequence of the fact that $[a, b]$ is connected, and g is continuous.)

Denote as before, by pr_x the projection $pr_x : \mathbb{R}^2 \to \mathbb{R}$ for which $pr_x(x, y) = x$ and note that by assumption $pr_x \circ \gamma : [0, 1] \to [0, x_1]$ is continuous and onto.

Pick now a sequence $s_n \to 0$ in $[0, 1]$.

Since $s_n \to 0$ and $pr_x \circ \gamma$ is continuous, we have $(pr_x \circ \gamma)(s_n) = x_n \to (pr_x \circ \gamma)(0) = 0$. Note that the sequence (s_n) may not work as the sequence (t_n) we are looking for, because the second coordinates $\sin \frac{1}{x_n}$ could be any number in $[-1, 1]$, so we may have the second coordinates converge to zero, in which case there is no contradiction.

So, for each $n \in \mathbb{N}$ consider the restriction $pr_x \circ \gamma : [0, s_n] \to [0, x_n]$. Pick $k_n \in \mathbb{N}$ such that $z_n = \frac{1}{\frac{\pi}{2} + k_n \pi} \in (0, x_n)$. This is possible, since $\frac{1}{\frac{\pi}{2} + k_n \pi} \to 0$ as $k_n \to \infty$. Moreover, we can pick k_n so that it has the same parity as n.

By the intermediate value theorem applied to $pr_x \circ \gamma$ on $[0, s_n]$, there is a $t_n \in (0, s_n)$ such that $(pr_x \circ \gamma)(t_n) = z_n = \frac{1}{\frac{\pi}{2} + k_n \pi}$.

Then we have $t_n \to 0$, since by construction $0 < t_n < s_n$ and $s_n \to 0$. Also, we have

$$\gamma(t_n) = (z_n, \sin \frac{1}{z_n}) = (\frac{1}{\frac{\pi}{2} + k_n \pi}, \sin (\frac{\pi}{2} + k_n \pi)) = (\frac{1}{\frac{\pi}{2} + k_n \pi}, (-1)^{k_n})$$

so that $\gamma(t_n) \not\to \gamma(0) = (0, 0)$ and that is a contradiction.

If $t' > 0$, we have $\gamma(t') = (0, b_0)$ for some $b_0 \in [-1, 1]$ and $\gamma(t) \in Y \ \forall t \in (t', 1]$.

We should then find a sequence $(t_n) \subset (t', 1]$ such that $t_n \to t'$, but $\gamma(t_n)$ is an alternating subsequence of $(x_m, (-1)^m)$, $x_m \to t'$, $m \in \mathbb{N}$ so that $\gamma(t_n) \not\to \gamma(t') = (0, b_0)$ which can be done similarly as in the previous case. ■

Theorem 0.0.14: Path-connectedness is a topological invariant: if X and Y are homeomorphic, then X is path connected if and only if Y is path connected.

This is a consequence of the fact that if $f : X \to Y$ is continuous and onto, and X is path-connected, then Y is path connected, too.