§76 Cutting and Pasting

To prove the classification theorem, we need to use certain geometric arguments involving what are called "cut-and-paste" techniques. These techniques show how to take a space X that is obtained by pasting together the edges of one or more polygonal regions according to some labelling scheme and to represent X by a different collection of polygonal regions and a different labelling scheme.

First, let us consider what it means to "cut apart" a polygonal region. Let P be a polygonal region with successive vertices $p_0, \ldots, p_n = p_0$, as usual. Given k with $1 < k < n - 1$, let us consider the polygonal regions Q_1, with successive vertices $p_0, p_1, \ldots, p_k, p_0$, and Q_2, with successive vertices $p_0, p_k, \ldots, p_n = p_0$. These regions have the edge $p_0 p_k$ in common, and the region P is their union.

Let us move Q_1 by a translation of $\mathbb{R}^2$ so as to obtain a polygonal region Q_1' that is disjoint from Q_2; then Q_1' has successive vertices $q_0, q_1, \ldots, q_k, q_0$, where q_i is the image of p_i under the translation. The regions Q_1' and Q_2 are said to have been obtained by ***cutting P apart*** along the line from p_0 to p_k. The region P is homeomorphic to the quotient space of Q_1' and Q_2 obtained by pasting the edge of Q_1' going from q_0 to q_k to the edge of Q_2 going from p_0 to p_k, by the positive linear map of one edge onto the other. See Figure 76.1.

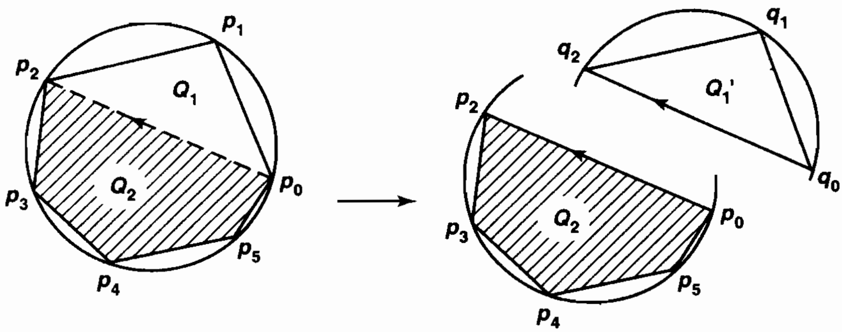

Figure 76.1

Now let us consider how we can reverse this process. Suppose we are given two disjoint polygonal regions Q_1' with successive vertices $q_0, \ldots, q_k, q_0$, and Q_2, with successive vertices $p_0, p_k, \ldots, p_n = p_0$. And suppose we form a quotient space by pasting the edge of Q_1' from q_0 to q_k onto the edge of Q_2 by p_0 to p_k, by the positive linear map of one edge onto the other. We wish to represent this space by a polygonal region P.

This task is accomplished as follows: The points of Q_2 lie on a circle and are arranged in counterclockwise fashion. Let us choose points $p_1, \ldots, p_{k-1}$ on this same circle in such a way that $p_0, p_1, \ldots, p_{k-1}, p_k$ are arranged in counterclockwise order, and let Q_1 be the polygonal region with these as successive vertices. There is a homeomorphism of Q_1' onto Q_1 that carries q_i to p_i for each i and maps the edge $q_0 q_k$ of Q_1' linearly onto the edge $p_0 p_k$ of Q_2. Therefore, the quotient space in question is homeomorphic to the region P that is the union of Q_1 and Q_2. We say that P is obtained by ***pasting Q_1' and Q_2 together*** along the indicated edges. See Figure 76.2.

Now we ask the following question: If a polygonal region has a labelling scheme, what effect does cutting the region apart have on this labelling scheme? More pre-

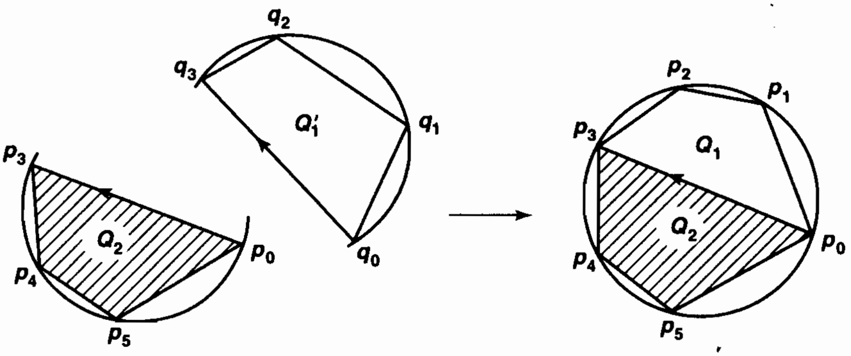

Figure 76.2

cisely, suppose we have a collection of disjoint polygonal regions $P_1, \ldots, P_m$ and a labelling scheme for these regions, say $w_1, \ldots, w_m$, where w_i is a labelling scheme for the edges of P_i. Suppose that X is the quotient space obtained from this labelling scheme. If we cut P_1 apart along the line from p_0 to p_k, what happens? We obtain $m + 1$ polygonal regions $Q_1', Q_2, P_2, \ldots, P_m$; to obtain the space X from *these* regions, we need one additional edge pasting. We indicate the additional pasting that is required by introducing a new label that is to be assigned to the edges q_0q_k and p_0p_k that we introduced. Because the orientation from p_0 to p_k is counterclockwise for Q_2, and the orientation from q_0 to q_k is clockwise for Q_1', this label will have exponent $+1$ when it appears in the scheme for Q_2 and exponent -1 when it appears in the scheme for Q_1'.

Let us be more specific. We can write the labelling scheme w_1 for P_1 in the form $w_1 = y_0y_1$, where y_0 consists of the first k terms of w_1 and y_1 consists of the remainder. Let c be a label that does not appear in any of the schemes $w_1, \ldots, w_m$. Then give Q_1' the labelling scheme y_0c^{-1}, give Q_2 the labelling scheme cy_1, and for $i > 1$ give the region P_i its old scheme w_i.

It is immediate that the space X can be obtained from the regions $Q_1', Q_2, P_2, \ldots, P_m$ by means of this labelling scheme. For the composite of quotient maps is a quotient map, so it does not matter whether we paste all the edges together at once, or instead paste the edge p_0p_k to the edge q_0q_k before pasting the others!

One can of course apply this procedure in reverse. If X is represented by a labelling scheme for the regions $Q_1', Q_2, P_2, \ldots, P_m$ and if the labelling scheme indicates that an edge of the first is to be pasted to an edge of the second (*and no other edge is to be pasted to these*), we can actually carry out the pasting so as to represent X by a labelling scheme for the m regions $P_1, \ldots, P_m$.

We state this fact formally as a theorem:

Theorem 76.1. *Suppose X is the space obtained by pasting the edges of m polygonal* *regions together according to the labelling scheme*

$$(*) \qquad y_0 y_1, w_2, \ldots, w_m.$$

Let c be a label not appearing in this scheme. If both y_0 and y_1 have length at least two, then X can also be obtained by pasting the edges of $m + 1$ polygonal regions together according to the scheme

$$(**) \qquad y_0 c^{-1}, c y_1, w_2, \ldots, w_m.$$

Conversely, if X is the space obtained from $m + 1$ polygonal regions by means of the scheme $(**)$, it can also be obtained from m polygonal regions by means of the scheme $(*)$, providing that c does not appear in scheme $(*)$.

Elementary operations on schemes

We now list a number of elementary operations that can be performed on a labelling scheme $w_1, \ldots, w_m$ without affecting the resulting quotient space X. The first two arise from the theorem just stated.

(i) *Cut.* One can replace the scheme $w_1 = y_0 y_1$ by the scheme $y_0 c^{-1}$ and $c y_1$, provided c does not appear elsewhere in the total scheme and y_0 and y_1 have length at least two.

(ii) *Paste.* One can replace the scheme $y_0 c^{-1}$ and $c y_1$ by the scheme $y_0 y_1$, provided c does not appear elsewhere in the total scheme.

(iii) *Relabel.* One can replace all occurrences of any given label by some other label that does not appear elsewhere in the scheme. Similarly, one can change the sign of the exponent of all occurrences of a given label a; this amounts to reversing the orientations of all the edges labelled "a". Neither of these alterations affects the pasting map.

(iv) *Permute.* One can replace any one of the schemes w_i by a cyclic permutation of w_i. Specifically, if $w_i = y_0 y_1$, we can replace w_i by $y_1 y_0$. This amount to renumbering the vertices of the polygonal region P_i so as to begin with a different vertex; it does not affect the resulting quotient space.

(v) *Flip.* One can replace the scheme

$$w_i = (a_{i_1})^{\epsilon_1} \cdots (a_{i_n})^{\epsilon_n}$$

by its formal inverse

$$w_i^{-1} = (a_{i_n})^{-\epsilon_n} \cdots (a_{i_1})^{-\epsilon_1}.$$

This amounts simply to "flipping the polygonal region P_i over.". The order of the vertices is reversed, and so is the orientation of each edge. The quotient space X is not affected.

(vi) *Cancel.* One can replace the scheme $w_i = y_0 a a^{-1} y_1$ by the scheme $y_0 y_1$, provided a does not appear elsewhere in the total scheme and both y_0 and y_1 have length at least two.

This last result follows from the three-step argument indicated in Figure 76.3, only one step of which is new. Letting b and c be labels that do not appear elsewhere in the total scheme, one first replaces $y_0aa^{-1}y_1$ by the scheme y_0ab and $b^{-1}a^{-1}y_1$, using the cutting operation (i). Then one combines the edges labelled a and b in each polygonal region into a single edge, with a new label. This is the step that is new. The result is the scheme y_0c and $c^{-1}y_1$, which one can replace by the single scheme y_0y_1, using the pasting operation (ii).

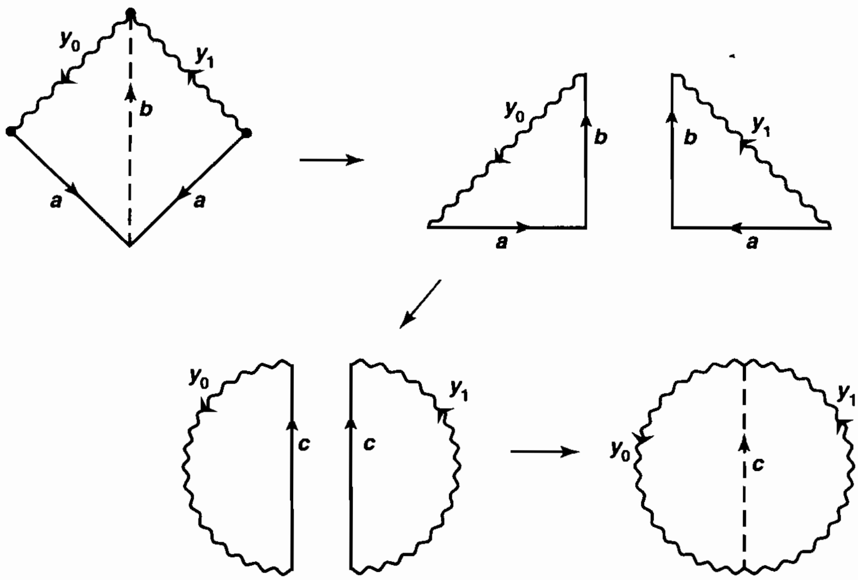

Figure 76.3

(vii) *Uncancel.* This is the reverse of operation (vi). It replaces the scheme y_0y_1 by the scheme $y_0aa^{-1}y_1$, where a is a label that does not appear elsewhere in the total scheme. We shall not actually have occasion to use this operation.

Definition. We define two labelling schemes for collections of polygonal regions to be ***equivalent*** if one can be obtained from the other by a sequence of elementary scheme operations. Since each elementary operation has as its inverse another such operation, this is an equivalence relation.

EXAMPLE 1. The Klein bottle K is the space obtained from the labelling scheme $aba^{-1}b$. In the exercises of §74, you were asked to show that K is homeomorphic to the 2-fold projective plane $P^2\#P^2$. The geometric argument suggested there in fact consists of the following elementary operations:

$aba^{-1}b \longrightarrow$	abc^{-1} and $ca^{-1}b$	by cutting
$\longrightarrow$	$c^{-1}ab$ and $b^{-1}ac^{-1}$	by permuting the first and flipping the second
$\longrightarrow$	$c^{-1}aac^{-1}$	by pasting
$\longrightarrow$	$aacc$	by permuting and relabelling.