Chapter 1

The fundamental group

We now discuss a new topological invariant: the fundamental group. Rather than studying invariants related to points as we did in point-set topology, this new invariant is based on studying (types of) loops on a surface.

More generally, we first need some machinery concerning "paths from point a to point b" of a surface. Loops are special - closed - paths, where $a = b$.

Precisely:

Definition 1.0.1: A path is a continuous mapping $\alpha : [0,1] \to X$ where (X, τ_X) is a topological space, such that $\alpha(0) = a$ is the beginning point and $\alpha(1) = b$ is the endpoint of the path.

Definition 1.0.2: We say that path α can be continuously deformed into β, or "α is **homotopic** to β" - denoted $\alpha \simeq \beta$ - if there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that

- $H(s,0) = \alpha(s)\ \forall s \in [0,1]$
- $H(s,1) = \beta(s)\ \forall s \in [0,1]$
- $H(0,t) = a = \alpha(0) = \beta(0)\ \forall t \in [0,1]$
- $H(1,t) = b = \alpha(1) = \beta(1)\ \forall t \in [0,1]$

The continuous mapping H is called a **homotopy from α to β**.

Remark 1.0.3: Suggestion: think of the variable s as determining location on the path α, β and other paths, and of the variable t as time.

That is: "H takes the path α to the path β in 1 unit of time"; more precisely, the family $f_s : [0,1] \to X$ does that, where $f_s(t) = H(s,t)\ \forall$ fixed $s \in [0,1]$.

On the other hand, $\forall$ fixed $t_0 \in [0,1]$, the mapping $\gamma : [0,1] \to X$ with $\gamma(s) = H(s,t_0)$ is an inbetween path.

Remark 1.0.4: In the literature there are variations on the types of homotopies used, depending on the purpose. The one above is a homotopy of paths *with fixed endpoints*. We will always compare paths with the same beginning and endpoints i.e. assume $\alpha(0) = \beta(0) = a$, $\alpha(1) = \beta(1) = b$ as well as consider only homotopies H with $H(0,t_0) = a = \alpha(0) = \beta(0)$ as well as $H(1,t_0) = b = \alpha(1) = \beta(1)$ $\forall$ inbetween paths $H(s,t_0)$ (i.e. $t_0 \in [0,1]$ fixed).

Example 1.0.5: (1.) In $\mathbb{R}^2$, all α, β paths as above are homotopic. We can use the **straight line homotopy** to show this:

$$H(s,t) = (1-t)\alpha(s) + t\beta(s) \ \ \forall s, t \in [0,1].$$

(2.) Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$. Take $\alpha, \beta, \gamma : [0,1] \to X$ with

$$\alpha(s) = (\cos(\pi s), \sin(\pi s))$$

$$\beta(s) = (\cos(\pi s), 2\sin(\pi s))$$

$$\gamma(s) = (\cos(\pi s), -\sin(\pi s)).$$

We have $\alpha \simeq \beta$, but α nor β is homotopic to γ since the "lack of origin" is in the way. We cannot continuously deform α or β *within* X to get γ while keeping their endpoints fixed, since we would have to pass over the missing origin. (A rigorous prove will be given later.)

To show α and β are homotopic, take $H(s,t) = (\cos(\pi s), t\sin(\pi s) + (1-t)2\sin(\pi s))$.

To show $\alpha \not\simeq \gamma$ and $\beta \not\simeq \gamma$ rigorously, we must develop more machinery.

(3.) Consider $\alpha, \beta : [0,1] \to \mathbb{R}^3$ be given by $\alpha(s) = (0, cos(\pi s), sin(\pi s))$ and $\beta(s) = (0, cos(\pi s), -sin(\pi s))$.

Then it is "intuitively clear", that $\alpha \simeq \beta$ in $X = S^2$, but $\alpha \not\simeq \beta$ in $X = S^1$ of the $\{x = 0\}$ i.e. yz-plane.

Remark 1.0.6: We denote that α and β are homotopic by $\alpha \simeq \beta$.

Proposition 1.0.7: $\simeq$ is an equivalence relation on the set of paths from a to b in X.

Proof. $\alpha \simeq \alpha$ by the homotopy $H(s,t) = \alpha(s)$. Next, if $\alpha \simeq \beta$ via the homotopy H, then we take F such that $F(s,t) = H(s, 1-t)$ to show $\beta \simeq \alpha$.

Lastly, to show transitivity, assume $\alpha \simeq \beta$ and $\beta \simeq \gamma$. Then if the homotopy H takes α to β and the homotopy F takes β to γ, then

$$G(s,t) = \begin{cases} H(s, 2t) & t \in [0, 1/2] \\ F(s, 2t-1) & t \in [1/2, 1] \end{cases}$$

takes α to γ "in one unit of time" i.e. $G(s,0) = \alpha(s)$ and $G(s,1) = \beta(s)$ for $s \in [0,1]$. Check that $G(0,t)$ and $G(1,t)$ "work the right way" i.e. $G(0,t) = \alpha(0) = \gamma(0)$ and $G(1,t) = \alpha(1) = \gamma(1)$. Also, G is well defined, since for $t = 1/2$ we have $H(s,t) = H(s,1) = \beta(s) = F(s,0) = F(s,t)$ and G is continuous by the Pasting Lemma, stated below.

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Lemma 1.0.8: Let $X = A \cup B$, where A, B are closed in X. Suppose $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ are continuous and such that $f(x) = g(x) \ \ \forall x \in A \cap B$.

Then the "piecewise defined" function

$$H(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is well-defined and continuous.

The proof is left as an exercise.

Remark 1.0.9: The equivalence class of α under $\simeq$ is denoted by $\langle \alpha \rangle$.

We will now define an operation on special paths.

Suppose we have two paths α and $\beta : [0, 1] \to X$ such that the second one starts where the first one ends i.e. $\alpha(1) = \beta(0)$. In this situation, we can define a new path by "running along the first one and then the second, twice as fast":

Definition 1.0.10: Given paths $\alpha, \beta : [0, 1] \to X$ such that $\alpha(1) = \beta(0)$, their **concatenation** is a new path

$$(\alpha * \beta)(s) := \alpha * \beta(s) = \begin{cases} \alpha(2s) & s \in [0, 1/2] \\ \beta(2s - 1) & s \in [1/2, 1] \end{cases}$$

Proposition 1.0.11: $*$ respects the equivalence classes under $\simeq$. That is, if $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$, then $\alpha * \beta \simeq \alpha' * \beta'$.

Remark 1.0.12: If we can show this, then the following definition makes sense: $\langle \alpha \rangle * \langle \beta \rangle = \langle \alpha * \beta \rangle$, whenever α and β are two paths that can be concatenated (i.e. if $\alpha(1) = \beta(0)$).

Proof. Let F be a homotopy taking α to α', and G be a homotopy taking β to β'. We want a new homotopy H taking $\alpha * \beta$ to $\alpha' * \beta'$. Let

$$H(s, t) := \begin{cases} F(2s, t) & s \in [0, 1/2] \\ G(2s - 1, t) & s \in [1/2, 1] \end{cases}$$

Note that H was obtained by concatenating, for each fixed time $t = t_0$, the "inbetween paths" $\gamma(s) = F(s, t_0)$ and $\delta(s) = G(s, t_0)$.

Exercise 1.0.13: Verify that $H(0, t)$, $H(1, t)$, $H(s, 0)$, $H(s, 1)$ "work the right way". Note, that H is continuous, by the Pasting Lemma.

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Now we are ready to define the fundamental group of a topological space (X, τ_X).

Fix a basepoint $x_0 \in X$. Recall that a **loop** is a path α for which $\alpha(0) = \alpha(1)$. Clearly, we can always concatenate two loops with the same base point.

Proposition 1.0.14: The set of equivalence classes of loops based at a point x_0 with the operation of concatenation $*$ is an algebraic group, which we denote $\pi_1(X, x_0)$.

Conjectures: Unit element: $\langle x_0 \rangle$, where $x_0 : [0,1] \to X$ maps $s \mapsto x_0 \ \ \forall s \in [0,1]$.

Inverses: Given $\langle \alpha \rangle$, we let $\langle \alpha \rangle^{-1} = \langle \alpha^{-1} \rangle$,where $\alpha^{-1}(s) = \alpha(1-s)$.

The proof that concatenation of equivalence classes of loops based at $x_0 \in X$ is associative, as well as that that the above candidates are indeed the unit and inverse elements consists of finding appropriate homotopies. This is discussed briefly in class.

Then we get that $\pi_1(X, x_0)$ is an algebraic group, indeed.

Exercise 1.0.15: Use straightline homotopy, to show that all loops $\alpha \simeq x_0$ for $x_0 \in \mathbb{R}^2$ so that $\Pi_1(\mathbb{R}^2, x_0) = 0$ ("it is trivial").

Just intuitively, give examples of other spaces with a trivial fundamental group.

Exercise 1.0.16: Find nonhomotopic loops and try to guess what the fundamental group of the following spaces is:

the cylinder $X = \{x^2 + y^2 = 1\} \subset \mathbb{R}^3$, the plane take-away a point, the torus, the plane take-away two points, the real projective plane.

1.0.1 Dependence on the basepoint.

Is the base point important, i.e. does the choice of x_0 in $\pi_1(X, x_0)$ matter? In general, yes. Let $X = S^1 \cup \{(3,0)\}$. If $x_0 = (3,0)$, then $\pi_1(X, x_0)$ is trivial, but it is not, if $x_0 \in S^1$.

However

Lemma 1.0.17: If X is path-connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

Proof: Since X is path-connected, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Then let $\phi : \pi(X, x_0) \cong \pi_1(X, x_1)$ be defined by $\phi(< \alpha >) =< \gamma^{-1} * \alpha * \gamma >$ and let $\psi : \pi_1(X, x_1) \cong \pi_1(X, x_0)$ be defined by $\phi(< \beta >) =< \gamma * \beta * \gamma^{-1} >$.

Check that ϕ and ψ are group homomorphisms, that is

$$\phi(< \alpha > * < \beta >) = \phi(< \alpha >) * \phi(< \beta >),$$

and a similar statement is true for ψ, as well as that $\phi \circ \psi = 1_{\pi(X,x_1)}$ and $\psi \circ \phi = 1_{\pi(X,x_0)}$.

These last two equations show that ϕ and ψ are bijections, so are in fact group isomorphisms. ■

From now on we assume that X is path-connected and (up to isomorphism) may use $\pi_1(X)$ for the fundamental group.

1.0.2 The fundamental group is a topological invariant.

We want to show that $\pi_1(X, x_0)$ is a topological invariant. That is: if $X \sim Y$ are homeomorphic, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ (group isomorphism). We need to find a way to pass from topology to algebra, and we do this by the "induced homomorphism."

Definition 1.0.18: Given a continuous map $f : X \to Y$, it "**induces a homomorphism of groups**,"

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)),$$

defined by $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$.

Proposition 1.0.19: f_* is well-defined.

Proof. We have to show that if $\alpha \cong \beta$ i.e. $<\alpha>=<\beta>$ then $f_*(<\alpha>= f_*(<\beta>)$.

Let $\alpha \cong \beta$ via the homotopy F. In particular, $F : [0,1] \times [0,1] \to X$. Then take $f \circ F : [0,1] \times [0,1] \to Y$ which is continuous, since it is a composition of continuous maps. Check that it deforms $f \circ \alpha$ to $f \circ \beta$.

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Proposition 1.0.20: f_* is a homomorphism.

Proof. One can check directly that

$$f_*(<\alpha> * <\beta>) = f_*(<\alpha>) * f_*(<\beta>)$$

$\forall <\alpha>, <\beta> \in \Pi_1(X, x_0)$

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The induced homomorphisms have the following properties:

Proposition 1.0.21: Let $g : X \to Y$, $f : Y \to Z$. For a topological space X and $x_0 \in X$, we have that $(\text{Id}_X)_* = 1_{\pi_1(X,x_0)}$ and $(f \circ g)_* = f_* \circ g_*$.

Proof. $(\text{Id}_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is defined by $\langle\alpha\rangle \mapsto \langle \text{Id}_X \circ \alpha\rangle = \langle\alpha\rangle$.

Next, we have $g : X \to Y$, $f : Y \to Z$ and by the definition of induced homomorphisms

$$\begin{aligned}(f \circ g)_* : \pi_1(X, x_0) &\to \pi_1(z, (f \circ g)(x_0)) \\ \langle\alpha\rangle &\mapsto \langle (f \circ g) \circ \alpha\rangle\end{aligned}$$

On the other hand, $f_* : \pi_1(Y, y_0) \to \pi_1(Z, f(y_0))$ takes $\langle\beta\rangle \mapsto \langle f \circ \beta\rangle$. We also have $g_* : \pi(X, x_0) \to \pi_1(Y, y_0)$. Then

$$(f_* \circ g_*)(\langle\alpha\rangle) = f_*(g_*(\langle\alpha\rangle)) = \langle f \circ (g \circ \alpha)\rangle.$$

Thus the left-hand-side $(f \circ g)_*$ is equal to the right-hand-side $f_* \circ g_*$ by associativity of composition.

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Now, we can finally check that $\pi_1(X, x_0)$ is a topological invariant. If $X \sim Y$, then there exist continuous $f : X \to Y$ and $g := f^{-1} : Y \to X$ such that $f \circ g = \text{Id}_Y$ and $g \circ f = \text{Id}_X$. For the induced homomorphisms, we then have

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

$$g_* : \pi_1(Y, y_0) \to \pi_1(X, g(y_0)).$$

We have $(f \circ g)_* = (\text{Id}_Y)_*$. By the previous proposition these imply that $f_* \circ g_* = 1_{\pi_1(Y,y_0)}$ and $g_* \circ f_* = 1_{\pi_1(X,x_0)}$. Therefore, f_* and g_* are bijections and so group isomorphisms.

Exercise 1.0.22: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2 \setminus \{p\}$.

Proof. Both are non-compact and all the topological invariants coming from point set topology that we considered fail to distinguish the two spaces.

However, the fundamental group of the first space is trivial, while that of the second space is non-trivial as the loop going around the missing point is not homotopic to the constant loop (one would have to move it over the missing point). This is "intuitively clear", a formal proof will be given later. ■

1.1 An application: Brouwer's fixed point theorem

Definition 1.1.1: Given $A \subset X$, a continuous, surjective map $r : X \to A$ is called a **retraction** if $r(a) = a$ for all $a \in A$. We also say that A is a **retract** of X.

Remark: Consider the inclusion map $i : A \to X$, $i(a) = a \ \forall a \in A$. If $r : X \to A$ is a retraction, we automatically have that, $r \circ i = \mathrm{Id}_A$. So, for the induced homomorphisms, $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$, we have $r_* \circ i_* = 1_{\pi_1(A,a_0)}$. Thus r_* has to be onto and i_* is 1-1.

Example 1.1.2: (1.) Consider the constant map $r : \mathbb{R}^2 \to \{(0,0)\}$. This is a retraction onto $A = \{(0,0)\}$.

(2.) $r : \{(x,y,z) \ : \ x^2 + y^2 = 1\} \to S^1 \subset \mathbb{R}^2$ where r is the projection $r(x,y,z) = (x,y)$. Then r is a retraction of the cylinder onto the circle.

Example 1.1.3: (a.) Give an example of a continuous map $f : \overline{D}^2 \to S^1$

(b.) Give an example of a continuous, onto map $f : \overline{D}^2 \to S^1$

(c.) Give an example of a continuous, onto map $f : \overline{D}^2 \to S^1$, that keeps S^1 fixed pointwise.

The last question above asks for a retraction $\overline{D}^2$ onto S^1. In fact, there is no such map, for if such a map $f = r$ existed, then $r_* \circ i_* = \mathrm{Id}_{S^1}$. This would mean that the composition $\pi_1(S_1) \to \pi_1(\bar{D}^2) \to \pi_1(S^1)$ would have to be the identity mapping on $\pi_1(S_1)$. But $\pi_1(\bar{D}^2)$ is trivial and $\pi_1(S^1) \cong \mathbb{Z}$, so this is a contradiction.

A very famous consequence of this fact is the **Brouwer fixed point theorem**:

Theorem 1.1.4: $\forall f : \overline{D}^2 \to \overline{D}^2$ continuous maps $\exists x \in \overline{D}^2$ with $f(x) = x$.

We proved this in class.