

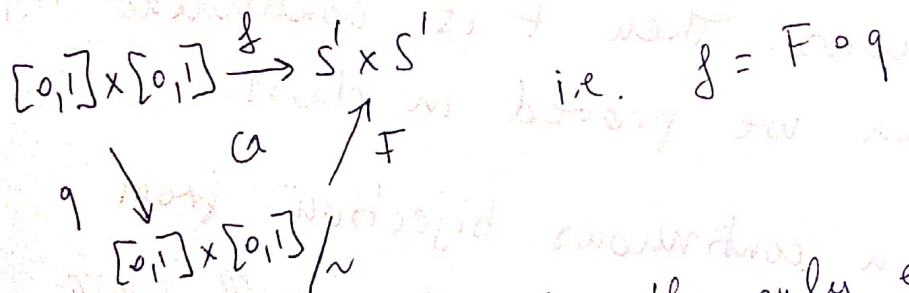
Claim: $[0,1] \times [0,1] / \sim \rightarrow S^1 \times S^1$
 $(x,0) \sim (x,1)$
 $(0,y) \sim (1,y)$

Pf: Consider $f: [0,1] \times [0,1] \rightarrow S^1 \times S^1$
 s.t. $(s,t) \mapsto (\cos 2\pi s, \sin 2\pi s, \cos 2\pi t, \sin 2\pi t)$

We know from calculus f is continuous, onto
 and $f(0,t) = f(1,t)$, $f(s,0) = f(s,1) \quad \forall t,s \in [0,1]$.

Now, define $F: [0,1] \times [0,1] / \sim \rightarrow S^1 \times S^1$
 s.t. $F([x,y]) = f((x,y))$ (here $[x,y]$ = equiv class of $(x,y) \in [0,1] \times [0,1]$)

i.e. so that the following diagram commutes:



1) Then F is well-defined: the only equiv classes in $[0,1] \times [0,1] / \sim$ that have more than one element are $\{(0,x), (1,x)\}$ and $\{(y,0), (y,1)\}$. One can check directly that $F([0,x]) = F([1,x])$, $F([y,0]) = F([y,1])$.

2) Also, since f is onto F is onto.

3) F is 1-1 since f is at most 2-to-1 and we glued the points of $[0,1] \times [0,1]$ which had the same image under f .

(glued = identified)

$\implies F$ is a bijection.

(4.) Note that $[0,1] \times [0,1]$ is compact, since it is closed & bounded in \mathbb{R}^2 , and the Heine-Borel theorem applies.

Since the quotient map $q: [0,1] \times [0,1] \rightarrow [0,1] \times [0,1] / \sim$ is onto, continuous the quotient is compact.

(5.) $S^1 \times S^1$ as subset of \mathbb{R}^4 is Hausdorff.

(~~The~~ Every subspace of a Hausdorff space is Hausdorff.)

(6.) f is continuous, so $F \circ q = f$ is also continuous. Then F is continuous by a lemma we proved in class.

So F is a continuous bijection from a compact space to a Hausdorff space, so it must be a homeomorphism.