Claim: $[0,1]\times[0,1] \Big/ {\substack{(x,0)\sim(x,1) \\ (0,y)\sim(1,y)}} \;\sim\; S^1 \times S^1$

Pf: Consider $f : [0,1]\times[0,1] \longrightarrow S^1 \times S^1$

s.t. $\quad (s,t) \longmapsto (\cos 2\pi s, \sin 2\pi s, \cos 2\pi t, \sin 2\pi t)$.

we know from calculus f is continuous, onto

and $f((0,t)) = f((1,t))$, $f((s,0)) = f((s,1))$ $\forall\, t, s \in [0,1]$.

Now, define $F : [0,1]\times[0,1]/\sim \;\longrightarrow S^1\times S^1$

s.t. $F([(x,y)]) = f((x,y))$ (here $[(x,y)]$ = equiv class of $(x,y) \in [0,1]\times[0,1]$)

i.e. so that the following diagram commutes:

$$\begin{tikzcd}
{[0,1]\times[0,1]} \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & S^1\times S^1 \\
{[0,1]\times[0,1]/\sim} \arrow[ur, "F"'] &
\end{tikzcd}$$

i.e. $f = F \circ q$.

1) Then F is well-defined: the only equiv classes in $[0,1]\times[0,1]/\sim$ that have more than one element are $\{(0,x), (1,x)\}$ and $\{(y,0),(y,1)\}$. One can check directly that $F([(0,x)]) = F([(1,x)])$, $F([(y,0)]) = F([(y,1)])$.

2) Also, since f is onto F is onto.

3) F is 1-1 since f is at most 2-to-1 and we glued the points of $[0,1]\times[0,1]$ which had the same image under f.
(glued = identified)

$\Longrightarrow$ F is a bijection.

(4.) Note that $[0,1] \times [0,1]$ is compact, since it is closed & bounded in $\mathbb{R}^2$, and the Heine-Borel theorem applies.

Since the quotient map $q: [0,1] \times [0,1] \to [0,1] \times [0,1]/\sim$ is onto, continuo[us]

the quotient is compact.

(5.) $S^1 \times S^1$ as subset of $\mathbb{R}^4$ is Hausdorff.
(Every subspace of a Hausdorff space is Hausdorff.)

(6.) f is continuous, so $F \circ q = f$ is also continuous. Then F is continuous by a lemma we proved in class.

So F is a continuous bijection from a compact space to a Hausdorff space, so it must be a homeomorphism.