To treat this pasting process formally requires some care. First, let us define precisely what we shall mean by a "polygonal region in the plane." Given a point c of $\mathbb{R}^2$, and given $a > 0$, consider the circle of radius a in $\mathbb{R}^2$ with center at c. Given a finite sequence $\theta_0 < \theta_1 < \cdots < \theta_n$ of real numbers, where $n \geq 3$ and $\theta_n = \theta_0 + 2\pi$, consider the points $p_i = c + a(\cos\theta_i, \sin\theta_i)$, which lie on this circle. They are numbered in counterclockwise order around the circle, and $p_n = p_0$. The line through p_{i-1} and p_i splits the plane into two closed half-planes; let H_i be the one that contains all the points p_k. Then the space

$$P = H_1 \cap \cdots \cap H_n$$

is called the ***polygonal region*** determined by the points p_i. The points p_i are called the ***vertices*** of P; the line segment joining p_{i-1} and p_i is called an ***edge*** of P; the union of the edges of P is denoted Bd P; and $P -$ Bd P is denoted Int P. It is not hard to show that if p is any point of Int P, then P is the union of all line segments joining p and points of Bd P, and that two such line segments intersect only in the point p.

Given a line segment L in $\mathbb{R}^2$, an ***orientation*** of L is simply an ordering of its end points; the first, say a, is called the ***initial point***, and the second, say b, is called the ***final point***, of the oriented line segment. We often say that L is oriented ***from a to b***; and we picture the orientation by drawing an arrow on L that points from a towards b. If L' is another line segment, oriented from c to d, then the ***positive linear map*** of L onto L' is the homeomorphism h that carries the point $x = (1-s)a + sb$ of L to the point $h(x) = (1-s)c + sd$ of L'.

If two polygonal regions P and Q have the same number of vertices, $p_0, \ldots, p_n$ and $q_0, \ldots, q_n$, respectively, with $p_0 = p_n$ and $q_0 = q_n$, then there is an obvious homeomorphism h of Bd P with Bd Q that carries the line segment from p_{i-1} to p_i by a positive linear map onto the line segment from q_{i-1} to q_i. If p and q are fixed points of Int P and Int Q, respectively, then this homeomorphism may be extended to a homeomorphism of P with Q by letting it map the line segment from p to the point x of Bd P linearly onto the line segment from q to $h(x)$. See Figure 74.1.

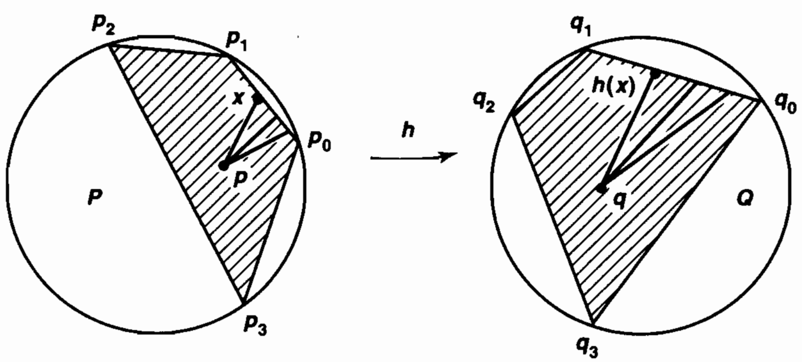

Figure 74.1

Definition. Let P be a polygonal region in the plane. A ***labelling*** of the edges of P is a map from the set of edges of P to a set S called the set of ***labels***. Given an orientation of each edge of P, and given a labelling of the edges of P, we define an equivalence relation on the points of P as follows: Each point of Int P is equivalent only to itself. Given any two edges of P that have the same label, let h be the positive linear map of one onto the other, and define each point x of the first edge to be equivalent to the point $h(x)$ of the second edge. This relation generates an equivalence relation on P. The quotient space X obtained from this equivalence relation is said to have been obtained by ***pasting the edges of P together*** according to the given orientations and labelling.

EXAMPLE 1. Consider the orientations and labelling of the edges of the triangular region pictured in Figure 74.2. The figure indicates how one can show that the resulting quotient space is homeomorphic to the unit ball.

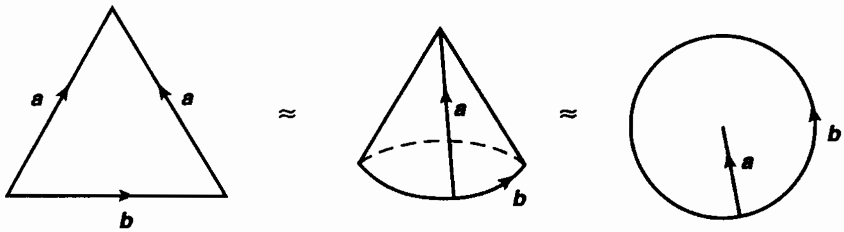

Figure 74.2

EXAMPLE 2. The orientations and labelling of the edges of the square pictured in Figure 74.3 give rise to a space that is homeomorphic to the sphere S^2.

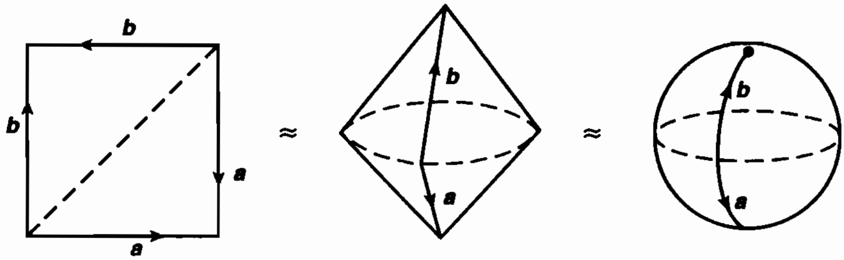

Figure 74.3

We now describe a convenient method for specifying orientations and labels for the edges of a polygonal region, a method that does not involve drawing a picture.

Definition. Let P be a polygonal region with successive vertices $p_0, \ldots, p_n$, where $p_0 = p_n$. Given orientations and a labelling of the edges of P, let $a_1, \ldots, a_m$ be the distinct labels that are assigned to the edges of P. For each k, let a_{i_k} be the label assigned to the edge $p_{k-1}p_k$, and let $\epsilon_k = +1$ or -1 according as the orientation assigned to this edge goes from p_{k-1} to p_k or the reverse. Then the number of edges of P, the orientations of the edges, and the labelling are completely specified by the symbol

$$w = (a_{i_1})^{\epsilon_1}(a_{i_2})^{\epsilon_2}\cdots(a_{i_n})^{\epsilon_n}.$$

We call this symbol a ***labelling scheme of length n*** for the edges of P; it is simply a sequence of labels with exponents $+1$ or -1.

We normally omit the exponents that equal $+1$ when giving a labelling scheme. Then the orientations and labelling of Example 1 can be specified by the labelling scheme $a^{-1}ba$, if we take p_0 to be the top vertex of the triangle. If we take one of the other vertices to be p_0, then we obtain one of the labelling schemes baa^{-1} or $aa^{-1}b$.

Similarly, the orientations and labelling indicated in Example 2 can be specified (if we begin at the lower left corner of the square) by the symbol $aa^{-1}bb^{-1}$.

It is clear that a cyclic permutation of the terms in a labelling scheme will change the space X formed by using the scheme only up to homeomorphism. Later we will consider other modifications one can make to a labelling scheme that will leave the space X unchanged up to homeomorphism.

EXAMPLE 3. We have already showed how the torus can be expressed as a quotient space of the unit square by means of the quotient map $p \times p : I \times I \to S^1 \times S^1$. This same quotient space can be specified by the orientations and labelling of the edges of the square indicated in Figure 74.4. It can be specified also by the scheme $aba^{-1}b^{-1}$.

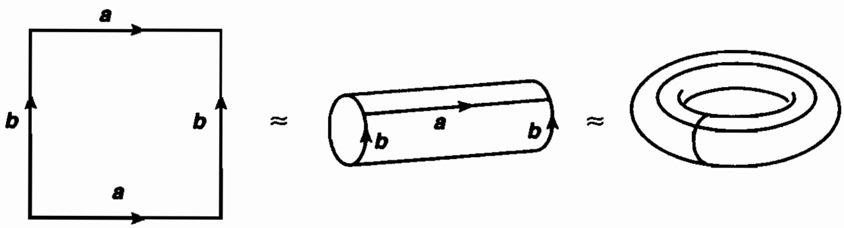

Figure 74.4

EXAMPLE 4. The projective plane P^2 is homeomorphic to the quotient space of the unit ball B^2 obtained by identifying x with $-x$ for each $x \in S^1$. Because the unit square is homeomorphic to the unit ball, this space can also be specified by the orientations and labelling of the edges of the unit square indicated in Figure 74.5. It can be specified by the scheme $abab$.

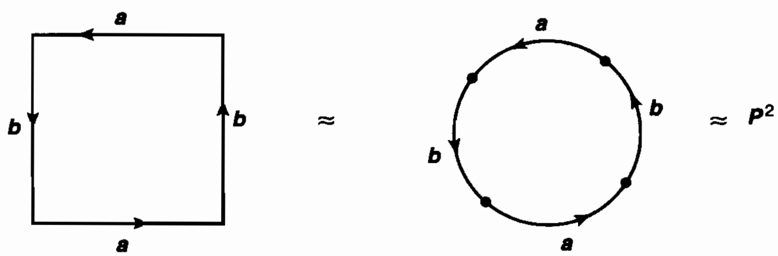

Figure 74.5

Now there is no reason to restrict oneself to a single polygonal region when forming a space by pasting edges together. Given a finite number $P_1, \ldots, P_k$ of disjoint polygonal regions, along with orientations and a labelling of their edges, one can form a quotient space X in exactly the same way as for a single region, by pasting the edges of these regions together. Also, one specifies orientations and a labelling in a similar way, by means of k labelling schemes. Depending on the particular schemes, the space X one obtains may or may not be connected.

EXAMPLE 5. Figure 74.6 indicates a labelling of the edges of two squares for which the resulting quotient space is connected; it is the space called the ***Möbius band***. Of course, this space could also be obtained from a single square by using the labelling scheme $abac$, as you can check.

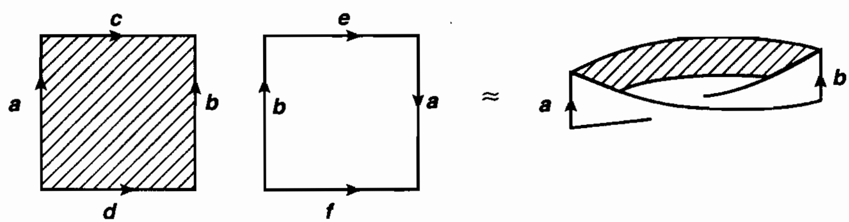

Figure 74.6

EXAMPLE 6. Figure 74.7 indicates a labelling scheme for the edges of two squares for which the resulting quotient space is not connected.

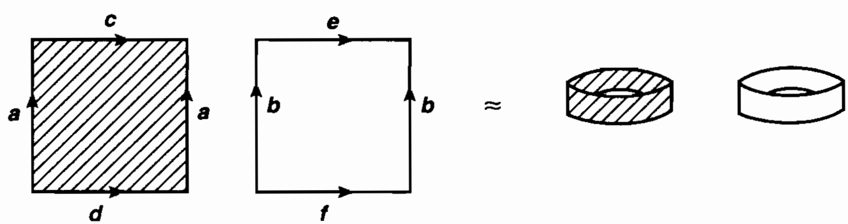

Figure 74.7