0.1 Methods to Calculate $\pi_1(X)$

0.1.1 Using deformation retractions

The idea here is that we can get rid of "unnecessary" chunks of space. Unnecessary - from the point of view of calculating the fundamental group. For example, with the cylinder $\{(x, y, z) \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^3$ it is intuitively clear that the fundamental group is determined by the number of times a loop goes around it and this is preserved if we contract the whole cylinder down to the circle $\{(x, y, 0) \,|\, x^2 + y^2 = 1\}$ - we . So how much a loop varies in the z direction is irrelevant from the point of view of the corresponding element of the fundamental group.

We now want to develop this intuitive idea precisely.

Definition 0.1.1: Given $A \subset X$, a continuous, surjective map $r : X \to A$ is called a **retraction** if $r(a) = a$ for all $a \in A$. We also say that A is a **retract** of X.

Remark: Consider the inclusion map $i : A \to X$, $i(a) = a \ \forall a \in A$. If $r : X \to A$ is a retraction, we automatically have that, $r \circ i = \mathrm{Id}_A$. So, for the induced homomorphisms, $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$, we have $r_* \circ i_* = 1_{\pi_1(A,a_0)}$. Thus r_* has to be onto and i_* is 1-1.

Example 0.1.2: (1.) Consider the constant map $r : \mathbb{R}^2 \to \{(0, 0)\}$. This is a retraction onto $A = \{(0, 0)\}$.

(2.) $r : \{(x, y, z) \ : \ x^2 + y^2 = 1\} \to S^1 \subset \mathbb{R}^2$ where r is the projection $r(x, y, z) = (x, y)$. Then r is a retraction of the cylinder onto the circle.

Example 0.1.3: (a.) Give an example of a continuous map $f : \overline{D}^2 \to S^1$

(b.) Give an example of a continuous, onto map $f : \overline{D}^2 \to S^1$

(c.) Give an example of a continuous, onto map $f : \overline{D}^2 \to S^1$, that keeps S^1 fixed pointwise.

The last question above asks for a retraction $\overline{D}^2$ onto S^1. In fact, there is no such map, for if such a map $f = r$ existed, then $r_* \circ i_* = \mathrm{Id}_{S^1}$. This would mean that the composition

$$\pi_1(S_1) \xrightarrow{i_*} \pi_1(\bar{D}^2) \xrightarrow{f_*} \pi_1(S^1)$$

should be the identity isomorphism on $\pi_1(S^1)$. But $\pi_1(\bar{D}^2)$ is trivial and $\pi_1(S^1) \cong \mathbb{Z}$, so this is impossible.

A very famous consequence of this fact is the **Brouwer fixed point theorem**:

Theorem 0.1.4: $\forall f : \overline{D}^2 \to \overline{D}^2$ continuous maps $\exists x \in \overline{D}^2$ with $f(x) = x$.

We proved this in class.

Now, consider the following

Definition 0.1.5: A is a **deformation retract** of X if there exists a homotopy $H : X \times [0,1] \to X$ map which is continuous such that

$$H(x,0) = x \quad \forall x \in X,$$

$$H(x,1) \in A \quad \forall x \in X,$$

$$H(a,t) = a \quad \forall t \in [0,1]$$

Note that this definition implies

- for each fixed $x \in X$, the mapping $\alpha : [0,1] \to X$, $\alpha(t) = H(x,t)$ is a path from x to some $a = H(x,1) \in A$ (so X can be deformed onto A along paths);
- the mapping $r : X \to A$, $r(x) = H(x,1)$ is a retraction of X onto A.

Example 0.1.6: 1. $X = \mathbb{R}^2$ can be deformation retracted onto $A = \underline{0}$ via

$$H((x,y),t) = (1-t)(x,y) = ((1-t)x, (1-t)y)$$

2. $X = \mathbb{R}^2$ can be deformation retracted onto $A = x$-axis$= x, 0$ via

$$H((x,y),t) = (x, (1-t)y)$$

3. $X = \mathbb{R}^3 \setminus \{(0,0)\}$ can be deformation retracted onto $A = S^2$ via

$$H(\underline{x},t) = (1-t)\underline{x} + t\frac{x}{||\underline{x}||}$$

4. $X = \mathbb{R}^3 \setminus \{z-\text{axis}\}$ can be deformation retracted onto the cylinder $A = \{(x,y,z) \,|\, x^2 + y^2 = 1\}$ via

$$H((x,y,z),t) = ((1-t)x + t\frac{x}{\sqrt{x^2+y^2}},\ (1-t)y + t\frac{y}{\sqrt{x^2+y^2}},\ z\,)$$

Finally, we also have the following lemma which shows that if a subset A is a deformation retract of a space X then they have the same, i.e. isomorphic fundamental groups.

Lemma 0.1.7: If $A \subset X$ and $a_0 \in A$ and A is a deformation retract of X, then

$$\pi_1(X,a_0) \cong \pi_1(A,a_0)$$

Proof. We know A is a deformation retract of X. This implies that $\exists r : X \to A$ retraction such that $r(x) = H(x,1)$ for some homotopy $H : X \times [0,1] \longrightarrow X$ with boundary conditions

$$H(x,0) = x \quad \forall x \in X,$$

$$H(x,1) \in A \quad \forall x \in X,$$

$$H(a,t) = a \quad \forall t \in [0,1]$$

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We claim that the induced homomorphism $r_* : \pi(X, a_0) \to \pi(A, a_0)$ is an isomorphism. Thus we have to show that r_* is onto and 1-1.

r_* is onto : let $< \alpha > \in \pi(A, a_0)$ ie $\alpha : [0, 1] \to A$ is a loop based at a_0 in A. Consider the inclusion map $i : A \to X$. Then $i \circ \alpha : [0, 1] \to X$ is a loop based at a_0 in X. Note that $r_*(< i \circ \alpha >) =< r \circ i \circ \alpha >=< \alpha >$, since $r \circ i = Id_A$.

r_* is 1-1 : First we show the following: let $\alpha : [0, 1] \to X$ be a loop based at a_0 in X. Then $r \circ \alpha : [0, 1] \to A$ is a loop based at a_0 in A and so $i \circ r \circ \alpha$ is a loop based at a_0 in X.
We have that α is homotopic to $i \circ r \circ \alpha$ via $G : [0, 1] \times [0, 1] \to X$ where

$$G(s, t) = H(\alpha(s), t)$$

Now, suppose for two loops $\alpha, \beta : [0, 1] \to X$ based at a_0 in A we have

$$r_*(< \alpha >) = r_*(< \beta >).$$

That means, by definition of induced homomorphisms, that

$$< r \circ \alpha >=< r \circ \beta >$$

so

$$r \circ \alpha \simeq r \circ \beta \ \in \text{ in } A.$$

Therefore

$$i \circ r \circ \alpha \simeq i \circ r \circ \beta \ \text{ in } X.$$

On the other hand, as shown above, we have

$$\alpha \simeq i \circ r \circ \alpha \in \text{ in } X$$

and

$$\beta \simeq i \circ r \circ \beta \ \in \text{ in } X.$$

So by transitivity,

$$\alpha \simeq \beta$$

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