Midterm Info

What to review:
– what a topological space, open/closed sets are, what it means for topological spaces to be homeomorphic and so continuous functions;
– the special topologies: subspace, product, quotient topologies, as well as finite, co-finite, co-countable topologies or topologies coming from a metric;
- the Mobius strip, $\mathbb{R}P^2$, the torus, Klein bottle, S^2 given in terms of identification diagrams (labelling words). You also need to know how to work with diagrams (cut-and-paste arguments).
– the Hausdorff property and convergence
- definition of compactness and the Heine-Borel theorem
– in addition, you need to review what boundary, interior and closure of a set are
– Review the examples we worked out for all of the above, as well as homeworks.

The midterm will be closed book, but you can prepare an A4 size paper cheat sheet for it. Write only one side; the other should be used for the second half of the course ie the final.

Some practice/review problems.

1. Decide if the following sets are compact:

   (i) $A = [0, 1]$ in $\mathbb{R}$ with the co-countable topology

   (ii) the quotient $D = \mathbb{R}/\sim$ where $x \sim y$ for $x, y \in \mathbb{R}$ if and only if both are rational or both are irrational

   (iii) $B = \{(x, y, z) \mid xy - z^5 = 0\}$, a subset of $\mathbb{R}^3$

2. **True or false?**

   (a) D^2 - the open unit disk in the plane is homeomorphic to $\overline{D}^2$ - the closed unit disk in the plane.

   (b) $D^2 \setminus \{p\}$ - the open unit disk in the plane minus a point is homeomorphic to $S^2 \setminus \{p, q\}$ - the sphere minus 2 points.

   (c) $GL_2(\mathbb{R})$ the set of 2x2 matrices with non-zero determinant is open in $\mathbb{R}^4$.

   (d) If X is compact and $F \subset X$ is a closed set, then F is compact.

   (e) If $f : X \to Y$ is a continuous bijection that is an open map then it is a homeomorphism.

   (f) $\partial(A) = \partial(cl(A))$

(g) $n \to 0$ as $n \to \infty$ if $\mathbb{R}$ has the arrow topology.

3. Consider the torus given as the appropriate identification space of $[0,1] \times [0,1]$. Cut the torus along what corresponds to the line segments joining the origin with $(\frac{1}{2}, 1)$ and the other joining $(\frac{1}{2}, 0)$ with $(1,1)$. What do you get?

4. What is wrong with the following argument?

   $\forall n, m \in \mathbb{Z}$ consider the intervals $U_n = (n, \infty)$, $W_m = (-\infty, m)$. Since each U_n, W_m is an open set when $\mathbb{R}$ has the usual topology and $\bigcup U_n \cup \bigcup W_m = \mathbb{R}$, we have that the collection $\mathsf{A} = \{U_n, W_m\}_{n,m \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$.

   Now, notice that already eg $W_3 \cup U_2 = (-\infty, 3) \cup (2, \infty) = \mathbb{R}$, so $\{U_2, W_3\}$ is a finite subcover of A.

   So $\mathbb{R}$ with the usual topology is compact.

5. What is wrong with the following argument?
   We show that
   $$cl(\cup A_\alpha) \subset \cup cl(A_\alpha):$$
   (i.e. find the mistake in the following argument)
   If $\{A_\alpha\}$ is a collection of sets in X and if $x \in cl(\cup A_\alpha)$, then every open set U that contains x intersects $\cup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in cl(A_\alpha)$.

   Is the statement nevertheless true?

6. What is wrong with the following argument?
   In a metric space (X, d) a unit ball S^1 around a point p (i.e. $S^1 = \{q \in X \mid d(p,q) = 1\}$) is closed, since it is the boundary of the open unit ball, and boundaries are always closed sets, as for any set A, one can show that $\partial A = cl(A) \cap cl(X \setminus A)$ (i.e. any boundary is the intersection of two closed sets).