We have to show

$$\mathcal{T} = \{(a, +\infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

is a topology on $\mathbb{R}$.

<u>Pf.)</u> Axiom 1 is clearly true.

To prove axiom 2, assume $U_1, U_2 \in \mathcal{T}$.

a) If U_1 or $U_2 = \emptyset$ (or both), then
$U_1 \cap U_2 = \emptyset \in \mathcal{T}$.

b) If U_1 or $U_2 = \mathbb{R}$, wlog assume $U_1 = \mathbb{R}$,
then $U_1 \cap U_2 = U_2 \in \mathcal{T}$.

c) Now, assume neither of U_1, U_2 is $\emptyset$ or $\mathbb{R}$.
Then $U_1 = (a_1, +\infty)$, $U_2 = (a_2, +\infty)$ for
some $a_1, a_2 \in \mathbb{R}$. WLOG assume $a_1 \leq a_2$.
Then $U_1 \cap U_2 = (a_1, +\infty) = U_1 \in \mathcal{T}$.

$\Rightarrow$ in all cases $U_1 \cap U_2 \in \mathcal{T}$, so axiom 2 holds.

Now, let $\{U_\lambda\}_{\lambda \in I} \subset \mathcal{T}$ be an arbitrary collection. We have to show that

$\bigcup_{\lambda \in I} U_\lambda \in \mathcal{T}$, so axiom 3 holds.

a) Assume first $U_\lambda = \phi \ \forall \lambda \in I$

Then $\bigcup U_\lambda = \phi$ and $\phi \in \tau$.

b.) Assume $\exists \lambda \in I$ s.t. $U_\lambda = \mathbb{R}$.

Then $\bigcup U_\lambda = \mathbb{R}$ and $\mathbb{R} \in \tau$.

c.) Assume now that $\exists \lambda \in I$ with $U_\lambda \neq \phi$ and $U_\lambda \neq \mathbb{R} \ \forall \lambda$.

So if $U_\lambda \neq \phi$ for some $\lambda \in I$,

then $U_\lambda = (a_\lambda, +\infty)$, where $a_\lambda \in \mathbb{R}$.

Consider then the set of left-endpoints

$\mathcal{A} = \{ a_\lambda \mid$ if $U_\lambda \neq \phi$ and $U_\lambda = (a_\lambda, +\infty) \}$,

Note that $\mathcal{A} \subset \mathbb{R}$ and let $b = \inf \mathcal{A}$.

<u>Claim:</u> $\bigcup_{\lambda \in I} U_\lambda = (b, +\infty)$

<u>Pf:</u> Since $b \leq a_\lambda \quad \forall \lambda \in I, U_\lambda \neq \phi$

we have $(b, +\infty) \supset U_\lambda$ for these λ

and so $(b, +\infty) \supset \bigcup_{\lambda \in I} U_\lambda$.

Now, assume $x \in (b, +\infty)$. That means $b < x$, which by the definition of b means that $\exists a_\lambda \in \mathcal{A}$ s.t. $b < a_\lambda < x$.

That means

$$x \in (a_\lambda, +\infty) = U_\lambda \text{ for that } \lambda$$

and so $x \in \bigcup_{\lambda \in I} U_\lambda$.

Since $x > b$ was arbitrary, we get

$$\bigcup_{\lambda \in I} U_\lambda \supset (b, +\infty).$$

So $\bigcup_{\lambda \in I} U_\lambda = (b, +\infty) \in \hat{\tau}$ and we are done.

(Note that $b = -\infty$ is possible, which means $\bigcup_{\lambda} U_\lambda = (-\infty, \infty) = \mathbb{R}$ in that case.

Also, $b = \inf A = \min A$ is possible if A is finite ($\Leftrightarrow$ $\{U_\lambda\}_{\lambda \in I}$ is finite $\Leftrightarrow$ I is finite).

In that case, $b = a_\lambda$ for some $a_\lambda \in I$.