Let X = some set, d = a metric on it.

<u>Claim</u> : X is open

<u>Pf</u>: $\forall x \in X$ and $r > 0$ we defined

$$B_x(r) = \{ y \in X \mid d(x,y) < r \}$$

So clearly, $B_x(r) \subset X$. Thus X is an open set.

Also, ϕ is open, since the requirement in the definition of open sets is vacuously true for it.

So the 1st axiom of topology holds.

<u>Claim</u>: If U, V are open subsets of X, then $U \cap V$ is also an open set.

<u>Pf</u>: We have to show that $\forall x \in U \cap V$

$\exists B_x(r) \subset U \cap V$,

Since $x \in U \cap V$, $x \in U$ which is an open set.

So $\exists s_1 > 0$ s.t. $B_x(s_1) \subset U$.

Also, $x \in V$ which is also open, so $\exists s_2 > 0$ s.t. $B_x(s_2) \subset V$. WLOG (without loss of generality), $s_1 \leq s_2$.

Then $B_x(s_1) \subset B_x(s_2)$, since if $y \in B_x(s_1)$ then $d(x,y) < s_1$, so $d(x,y) < s_2$, so $y \in B_x(s_2)$.

Thus $B_x(s_1) \subset V$.

$\Rightarrow B_x(s_1) \subset U \cap V$ and $r = s_1$ is a good choice.

$\Rightarrow U \cap V$ is an open set.

Thus Axiom 2 of topology is satisfied.

<u>Claim</u>: If $\{U_\lambda\}$ is an arbitrary collection of open sets, then $\bigcup_\lambda U_\lambda$ is open as well.

<u>Proof</u>: Let $x \in \bigcup_\lambda U_\lambda$, where $\lambda \in I$, some random index set.

Then $\exists \lambda_0 \in I$ s.t. $x \in U_{\lambda_0}$.

Since U_{λ_0} is an open set $\exists r > 0$ s.t. $B_x(r) \subset U_{\lambda_0}$. But then $B_x(r) \subset \bigcup_\lambda U_\lambda$

$\Rightarrow \bigcup_\lambda U_\lambda$ is an open set and axiom 3 holds.