Let (X, d) be a metric space.

Let $p \in X$, $r > 0$.

We want to show

$\overline{B_p}(r) = \{q \in X \mid d(p,q) \leq r\}$ is a closed set in the topology generated by d.

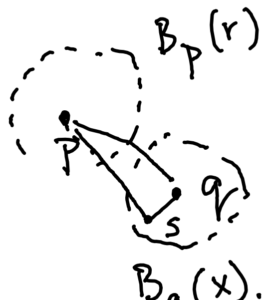
That is $X \setminus \overline{B_p(r)} = \{q \in X \mid d(p,q) > r\}$ is an open set.

So pick $q \in X \setminus \overline{B_p}(r)$

$\Rightarrow d(p,q) > r$.

Let $x = d(p,q) - r > 0$,

and consider $B_q(x)$.



<u>Claim:</u> $B_q(x) \subset X \setminus \overline{B_p(r)}$

<u>Pf:</u> Let $s \in B_q(x)$, so $d(q,s) < x$.

By the triangle inequality

$d(p,s) + d(q,s) \geq d(p,q) \Rightarrow$

$d(p,s)+x > d(p,q)$

Since $x = d(p,q) - r$,

we have

$d(p,s)+x > x+r$

so $d(p,s) > r \Rightarrow s \in X \setminus \overline{B_p(r)}$

Since $s \in B_q(x)$ was arbitrary,

we have $B_q(x) \subset X \setminus \overline{B_p(r)}$

so $X \setminus \overline{B_p(r)}$ is open $\Rightarrow$

$\overline{B_p(r)}$ is closed.