Let $\{U_i\}_{i=1}^{n} \in \tau_X$ a finite collection, ~~not all of as~~

Then $X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i)$ by the de-Morgan laws.

Since the $U_i \in \tau_X$ we have $X \setminus U_i$ finite $\forall i$

OR $\exists i$ s.t. $U_i = \phi$ and X is infinite.

In the first case $\bigcup_{i=1}^{n} (X \setminus U_i)$ is the finite union of finite many sets, so is finite.

(Clearly, $\left| \bigcup_{i=1}^{n} (X \setminus U_i) \right| \leq \sum_{i=1}^{n} |X \setminus U_i|$ where $|V|$ = cardinality of V)

~~In the second case,~~

Then we have $X \setminus \bigcap_{i=1}^{n} U_i$ is finite so $\bigcap_{i=1}^{n} U_i \in \tau_X$

In the second case, $\bigcap_{i=1}^{n} U_i = \phi \in \tau_X$.

If $\{U_\lambda\}_\lambda$ is an arbitrary collection of open sets in τ_X, then let $W = \bigcup_\lambda U_\lambda$.

$X \setminus W = X \setminus \bigcup_\lambda U_\lambda = \bigcap_\lambda (X \setminus U_\lambda)$ by the de-Morgan laws

Clearly, $\left| \bigcap_\lambda (X \setminus U_\lambda) \right| \leq |X \setminus U_\lambda| \ \forall \lambda$, in particular $\bigcap_\lambda (X \setminus U_\lambda)$ is finite, so $W \in \tau_X$.

Thus τ_X is a topology.

4.b. Let $g: X = \mathbb{N} \to Y = \mathbb{N}$ s.t. $\tau_X = \tau_Y$ = co-finite top

$$g(x) = \begin{cases} 2 & \text{if } x \text{ is even} \\ 4 & \text{if } x \text{ is odd} \end{cases}$$

Then g is not continuous.
To show this we must find $V \in \tau_Y$
s.t. $g^{-1}(V) \notin \tau_X$.
Consider $V = \mathbb{N} \setminus \{4\}$. Then $V \in \tau_Y$, since
$\mathbb{N} \setminus V = \{4\}$ is finite.
But $g^{-1}(V) = \{\text{all evens}\} \notin \tau_X$,
since $\mathbb{N} \setminus g^{-1}(V) = \{2k+1\}_{k \in \mathbb{N}}$ = all odds is not finite