#3. Given $(X, \mathcal{T}_X)$ and $A \subset X$, have to show that $V \subset A$ is closed in $A \Leftrightarrow \exists W$ closed in $X$ s.t.
$$V = W \cap A$$
(Here $A$ has the subspace topology of $\mathcal{T}_X$.)

Pf. $\Rightarrow$ Suppose $V$ is closed in $A$.
Then $A \setminus V$ is open in $A$.
By the definition of subspace topology, this means $\exists U \in \mathcal{T}_X$ s.t. $A \setminus V = U \cap A$.
Let $W = X \setminus U$. Then this is a closed set in $X$.

Claim: $V = W \cap A$.

We know $A \setminus V = U \cap A$, so
$$V = A \setminus (A \setminus V) = A \setminus (U \cap V)$$
By the de Morgan Laws $A \setminus (U \cap V) = (A \setminus U) \cup (A \setminus A)$

So $V = A \setminus U$. But $A \setminus U = (X \setminus U) \cap A$ since both ~~are~~ contain those elements in $A$ that are not in $U$.

$\Leftarrow$ Suppose $V = W \cap A$ for some $W$ closed in $X$.
Want to show $W$ is closed in $A$, that is
$A \setminus V$ is open in $A$, that is $\exists U \in \mathcal{T}_X$ s.t. $A \setminus V = U \cap A$
Let $\underbrace{U = X \setminus W}_{\text{so } U \in \mathcal{T}_X}$, We know $V = W \cap A$.

Then $A \setminus V = A \setminus (W \cap A) = (A \setminus W) \cup (A \setminus A) = A \setminus W$

But $A \setminus W = (X \setminus W) \cap A$.

So $A \setminus V = (X \setminus W) \cap A = U \cap A$ and we are done.