1 Basic notions, topologies

MAIN GOAL: We want to study "shapes". In particular, we want to be able to decide when two "shapes" X and Y are "the same".

More precisely, suppose that $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$.

Definition 1 *X and Y are "topologically the same, equal", if there exists a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.*

Thus for X and Y to be topologically the same, they have to be equal at least as sets i.e. they at least have to have the same cardinality. The last part of the following exercise provides a useful tool for checking that a function is bijective.

Exercise 2 *Suppose the mappings $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f = Id_X$ (where Id_X is the identity mapping on X, that is $Id_X(p) = p \ \forall p \in X$). Show that f has to be 1-1 and g onto.*

Provide examples to show that f does not have to be onto and g does not have to be 1-1.

Conclude that if $g \circ f = Id_X$ and $f \circ g = Id_Y$ then f and g are bijections, $g = f^{-1}$ and X and Y have the same cardinality.

(Note that the above exercise applies to any maps f, g.)

TERMINOLOGY, NOTATION: f as in definition 1 is called a **"homeomorphism"** and we say that X **and** Y **are "homeomorphic"**, if there is a homeomorphism between them. This is denoted as $X \sim Y$.

So, our aim is to work out methods to decide when two "shapes" ie subsets of Euclidean spaces - in full generality, when two topological spaces - are homeomorphic.

WARNING: If f is a continuous bijection, that does not imply that f^{-1} would be continuous as well. For example, the parametrization $f : [0, 1) \to (\cos 2\pi t, \sin 2\pi t)$ is continuous and 1-1, onto the circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$, whose inverse is not continuous.

EXAMPLES FROM CALCULUS:

1. $(0,1) \sim \mathbf{R}$ and in fact $\sim (a,b)$ i.e. all open intervals of $\mathbf{R}$ are homeomorphic.
   Here, we use the fact, that $\sim$ is an equivalence relation.

2. circle $\sim$ a square

3. the cylinder $\{(x,y,z)|x^2+y^2=1\}$ is homeomorphic to the hyperboloid $\{(x,y,z)|x^2+y^2=1+z^2\}$

4. CLASSICAL EXAMPLE: $S^2 \setminus \{p\} \sim \mathbf{R}^2$. (**stereographic projection**).

5. In fact, $S^n \setminus \{p\} \sim \mathbf{R}^n$ for any dimension n.

RELATED EXERCISE:

1. Show explicitly that the following three spaces are homeomorphic
   a.) the (open) cylinder
   b.) plane minus a point
   c.) the open annulus (e.g. $\{(x,y,0)\,|\,1 < x^2+y^2 < 4\}$).

1.1 Precise formulation of "shapes": topological spaces.

Very exactly, instead of "shapes" we will work with "topological spaces".

Definition 3 *A "topological space" is a pair (X, τ_X) that consists of a set X and a prescribed collection of subsets of X denoted by τ_X (that is $\tau_X \subset \mathcal{P}(\mathcal{X})$, where $\mathcal{P}(\mathcal{X})$ denotes the power set of X).*

The collection τ_X must satisfy the following properties:

1. *$\emptyset \in \tau_X$ and $X \in \tau_X$*

2. *If $U_\alpha \in \tau_X$ for $\alpha \in I$ an* arbitrary *index set, then $\cup_\alpha U_\alpha \in \tau_X$.*

3. *If $U_1, U_2 \in \tau_X$ then $U_1 \cap U_2 \in \tau_X$.*

Exercise 4 *Show that axiom 3 may be replaced by axiom 3':
"If $U_1, U_2, ..., U_n \in \tau_X$ then $U_1 \cap U_2 \cap ... \cap U_n \in \tau_X$".*

That is, axiom 3 is true if and only if axiom 3' is true.

Terminology: we call the collection τ_α "a topology" on X. The elements of τ are referred to as "open sets".

EXAMPLES OF TOPOLOGIES:

1. Let X be any set and $\tau = \{\emptyset, X\}$. This is the so called "antidiscrete topology".
2. Let X be any set and $\tau = \mathcal{P}(\mathcal{X})$, the power set of X. This is the so called "discrete topology".
3. Let $X = \{a, b, c\}$. Then $\tau_1 = \{\emptyset, X, \{a\}\}$ or $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$ are topologies on X, while $\tau_3 = \{\emptyset, X, \{a\}, \{b\}\}\}$ is not.
   τ_1 and τ_2 are examples of "finite topologies".
4. Let $X = \mathbf{R}$. The collection $\tau_1 = \{\emptyset, X, \{(a, +\infty)\}_{a \in \mathbf{R}}\}$ is a topology on X, while $\tau_2 = \{\emptyset, X, \{[a, +\infty)\}_{a \in \mathbf{R}}\}$ is not. The topology τ_1 is sometimes called the "arrow-topology".

RELATED EXERCISES:

1. Let $X = \{a, b, c\}$. Consider the collection of subsets of X where $\mathcal{A} = \{\emptyset, \{b, c\}\}$. Include extra subsets in $\mathcal{A}$ to make it into a topology. Experiment with $\mathcal{A} = \{\emptyset, \{a\}, \{b, c\}\}$.
2. Is $\tau = \{\emptyset, \mathbf{R}\} \cup \{(-\frac{1}{2^n}, \frac{1}{2^n})\}_{n \in \mathbf{N}}$ a topology on $\mathbf{R}$?
3. For an infinite set X, fix $p \in X$. Consider those subsets of X that do not contain p. Show that these sets, together with X, form a topology on X.
4. For an infinite set X consider the collection of subsets

   $$\tau = \{\emptyset, X\} \cup \{U \subset X \mid X \setminus U \text{ is finite }\}$$

   .

   Show that τ is a topology on X. (It is called the *co-finite* topology on X.)

1.2 The usual (or standard) topology on $\mathbf{R}^n$.

In calculus one learns about open intervals (a, b), $-\infty < a < b < \infty$ of the real line $\mathbf{R}$ very early. These are special (1-dimensional) cases of *open disks* of $\mathbf{R}^2$, or even more generally, of *open balls* of $\mathbf{R}^n$.

Definition 5 *Let $\underline{x} = (x_1, x_2, ..., x_n)$ denote points of $\mathbf{R}^n$.*

An open ball centered at $\underline{x} \in \mathbf{R}^n$, of radius r is

$$B_{\underline{x}}(r) = \{\underline{y}\,|d(\underline{x}, \underline{y}) < r\}$$

where $d(\underline{x}, \underline{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

i.e. the open ball consists of all those points $\underline{y}$ of $\mathbf{R}^n$ that are closer to $\underline{x}$ than r.

Using open balls, one can then consider "open sets" which are defined as follows:

Definition 6 *A set U of $\mathbf{R}^n$ is called an "open set" if for every point of it, you can find an open ball centered at that point such that the ball is entirely inside U.*

That is: U is an open set of $\forall \underline{x} \in U$, $\exists B_{\underline{x}}(r)$ for some $r > 0$ such that $B_{\underline{x}}(r) \subset U$.

Definition 7 *A set is called "closed" if its complement is "open".*

See more on the motivation for this

EXERCISES:

1. Are the following sets open/closed?
   a.) The interval $(1, 2)$ in $\mathbf{R}$. The line segment $(1, 2) \subset \mathbf{R}$ viewed in $\mathbf{R}^2$.
   b.) The interval $[2, 3]$ in $\mathbf{R}$. The line segment $[2, 3] \subset \mathbf{R}$ viewed in $\mathbf{R}^2$.
   c.) $\cap_{i=1}^{\infty}[-1, 1/n)$ in $\mathbf{R}$.
   d.) $\mathbf{R}^n$ in $\mathbf{R}^n$
   e.) $\{r \in (0, 1)|r \in \mathbf{Q}\}$ in $\mathbf{R}$.

f.) $\{(x,y) \in \mathbf{R}^2 \,|\, 0 < x \leq 1\}$
g.) $\{(x,y) \in \mathbf{R}^2 \,|\, |x| = 1\}$
h.) $\{\frac{1}{n} \,|\, n \in \mathbf{N}\}$

2. a.) Prove the de-Morgan Laws

$$X \setminus \cup U_\alpha = \cap(X \setminus U_\alpha)$$

and

$$X \setminus \cap U_\alpha = \cup(X \setminus U_\alpha)$$

b.) Use these to show, that in a topology an arbitrary intersection and finite union of closed sets is closed.

c.) Give examples of open sets $\{U_\alpha\}$ (in any space you like and collection of your choice) for which $\cap_\alpha U_\alpha$ is i.) open ii.) closed, iii.)neither.

3. Show that an open disc of $\mathbf{R}^2$ is an open set. Work out a proof that can be generalized to show: any open ball of $\mathbf{R}^n$ is an open set.

4. Show that an arbitrary union and finite intersection of open sets of $\mathbf{R}^n$ is an open set.

Consider now the set of all open sets of $\mathbf{R}^n$,

$$\tau = \{U \subset \mathbf{R}^n \,|\, U \text{ is open in } \mathbf{R}^n\}$$

Using the fact that $\mathbf{R}^n$ and $\emptyset$ are also open sets, the last exercise above shows, that τ forms a topology on $\mathbf{R}^n$ – and this is the "usual" or "standard" topology on $\mathbf{R}^n$.

1.3 The topology generated by a metric.

Now, we generalize the previous example to any set X on which one can measure the distance between points. More precisely,

Definition 8 *Given a set X, we say that the non-negative map $d : X \times X \to \mathbf{R}_{\geq 0}$ is a* **metric** *(or distance function) on X if*
a.) $\forall x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$
b.) $\forall x, y \in X$ we have $d(x, y) = d(y, x)$
c.) $\forall x, y, z \in X$ we have $d(x, y) \leq d(x, z) + d(z, y)$. (This is the triangle inequality.)

TERMINOLOGY: The pair (X, d) is called a "metric space".

EXAMPLES OF METRICS ON $\mathbf{R}^2$
Let $\mathbf{x} = (x_1, y_1)$, $\mathbf{y} = (x_2, y_2) \in \mathbf{R}^2$.

- the usual metric $d_1(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- $d_2(\mathbf{x}, \mathbf{y}) = |x_1 - x_2| + |y_1 - y_2|$
- $d_3(x, y) = max\{|x_1 - x_2|, |y_1 - y_2|\}$
- the discrete metric $d_4(x, y) = 0$ if $\mathbf{x} = \mathbf{y}$ otherwise $d(\mathbf{x}, \mathbf{y}) = 1$

EXERCISES:
1. Check that the above are metrics indeed.
2. Sketch the unit circle centered at the origin, in each case.
3. Remember: the formula for the discrete metric works for any set X, therefore any set can be equipped with a metric.

In this setting, i.e. given a set X and a metric d on it, the notion of **open balls** and of **open sets** can be defined analogously to the case of $\mathbf{R}^n$:

Definition 9 *Let $\mathbf{x}, \mathbf{y}$ denote points of X.*

An open ball centered at $\mathbf{x} \in X$, of radius r is

$$B_{\mathbf{x}}(r) = \{\mathbf{y}\,|d(\mathbf{x}, \mathbf{y}) < r\}.$$

i.e. the open ball consists of all those points $\mathbf{y}$ of X that are closer to $\mathbf{x}$ than r.

EXERCISES:

1. Consider a set X with the discrete metric. Let $p \in X$ be an arbitrary point. What are the open balls $B_p(\frac{1}{2})$, $B_p(1)$, $B_p(2)$?
2. a.) Suppose d is a metric on a set X Check that the pair (X, d') where $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ for all $x, y \in X$ is also a metric space.

   b.) If d is the usual metric on $X = \mathbf{R}^2$, then describe open balls centered at the origin with respect to the metric d'.

Using open balls, one can then consider "open sets" which are defined as follows:

Definition 10 *A set U of X is called an "open set" if for every point of it, you can find an open ball centered at that point such that the ball is entirely inside U.*

That is: U is an open subset of X if $\forall \mathbf{x} \in U$, $\exists B_y(r)$ for some $r > 0$ such that $B_{\mathbf{x}}(r) \subset U$.

EXERCISE:

1. Show that given a set X and a metric d on it, still, the union of arbitrary many open sets is an open set and also the intersection of finite many open sets is an open set.

Since X and $\emptyset$ are also open sets, this last exercise shows, that **the open sets of X that were defined using open balls i.e. the metric d, form a topology – this is called the "topology generated by the metric d" on X.**

EXERCISES:

1. Show that in this setting too, open balls are open sets.
2. Think through: the discrete metric generates the discrete topology.
3. Give an argument that shows, metrics d_2 and d_3 above (of the first examples of metrics on $\mathbf{R}^2$) also generate the usual topology on $\mathbf{R}^2$.
4. Give an argument that shows: if a set X has more than 2 points then the anti-discrete topology on it is not metrizable (i.e. there is no metric that would generate this topology)).

1.4 Terminology - abstraction jump.

Motivated by the case of the usual and metric topologies, the sets of *any* topology are referred to as "open sets", even for topologies that have nothing to do with open balls (i.e. are not generated by a metric).

So people would, for example, say: "the only open sets in the anti-discrete topology are the empty set and the entire set X itself".

And you hear: "in a topology, the union of arbitrary many open sets is open". Well, yes, otherwise we would not call it a topology.

Nevertheless, this is a fact to be verified in case of topologies generated by a metric, where open sets are defined via open balls. And in this case, one says: *since* the open sets of a metric space satisfy the arbitrary union requirement (and the other requirements), they form a topology.

Also,

Definition 11 *Given a set X and a (ie any) topology τ on it, a set $V \subset X$ is called* a closed set *if its complement is open ie $X \setminus V \in \tau$.*

2 Continuity

Definition 12 *Given two topological spaces (X, τ_X) and (Y, τ_Y) and a function $f : X \to Y$, we say that f is continuous if for all $V \in \tau_Y$ we have $f^{-1}(V) \in \tau_X$.*

To clarify: by definition, for a function $f : X \to Y$ and $V \subset Y$ the set $f^{-1}(V) \subset X$ is defined by $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$.

NOTE: Since we call sets of τ_X and τ_Y "open sets" (in X resp. Y) this definition can be worded as:

A function (between topological spaces) is continuous if the preimage of every open set is an open set.

SOME RELATED EXERCISES:

1. Consider $X = \mathbf{R}$ and the topologies

(a) $\tau_1 = \{\emptyset, \mathbf{R}\} \cup \{(-\frac{1}{2^n}, \frac{1}{2^n})\}_{n\in\mathbf{N}}$

(b) $\tau_3 = \mathcal{P}(X)$ discrete topology

(c) $\tau_4 = \{\emptyset, \mathbf{R}\}$ anti-discrete topology

(d) $\tau_5 =$ the usual topology $=$ the topology determined by the usual (Eucleidean) metric

(e) $\tau_6 = \{\emptyset, \mathbf{R}\} \cup \{(a, \infty)\}_{a\in\mathbf{R}}$

Question1: For each of the topologies decide if the intervals $(3, 7)$ $(3, \infty)$ $[3, 7]$ $[3, \infty)$ are open, closed, both or neither.

Question2: Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x$. Is $f : (\mathbf{R}, \tau_2) \to (\mathbf{R}, \tau_3)$ continuous? How about $f : (\mathbf{R}, \tau_3) \to (\mathbf{R}, \tau_2)$? Or $f : (\mathbf{R}, \tau_3) \to (\mathbf{R}, \tau_6)$?

Experiment with other combinations.

2. Think through: given a map $f : X \to Y$, if X is equipped with the discrete topology, or Y is equipped with the anti-discrete topology, then f is continuous.

3. Let $f : X \to Y$ be the constant map, i.e. $f(x) = a$ for all $x \in X$ and some $a \in Y$. Show that f is continuous, for all topologies τ_X on X and τ_Y on Y.

4. Let $X = \mathbf{N}$ and consider $g : \mathbf{N} \to \mathbf{N}$ given by

$$g(x) = \begin{cases} 2 & \text{if } x \text{ is even} \\ 4 & \text{if } x \text{ is odd} \end{cases}$$

If $\mathbf{N}$ is equipped with the co-finite topology both as a source (domain) and target (range), is g continuous?

5. Suppose $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f$.

a.) Think through: if f and g are continuous, then h is continuous.

b.) If f and h are continuous, must g be continuous?

c.) If f and g are continuous, must h be continuous?

The definition of continuity, as given above, is motivated by the following lemma, which also shows that if a function f was continuous in the "calculus or real-analytic sense" then it is still continuous in the general topological sense:

The lemma is stated for a function of one variable, but is true for mappings between any dimensional spaces.

Lemma 13 *A mapping* $f : \mathbf{R} \to \mathbf{R}$ *is continuous if and only if for all* $V \subset \mathbf{R}$ *open sets (open in the usual topology) we have* $f^{-1}(V)$ *open in* $\mathbf{R}$.

Recall that $f : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ if it is continuous at every point $a \in \mathbf{R}$, i.e. the left hand side of the statement is "local" in nature.

Also, by definition, f is continuous at $a \in \mathbf{R}$ if $\forall \epsilon > 0 \ \exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ if $|x - a| < \delta$. So the definition of continuity and thus the left hand side of the lemma depends directly on "distance" i.e. how far $f(x)$ is from $f(a)$ depending on the distance between x and a.

On the other hand, the right hand side uses only the notion of open sets, it provides a "global view" of continuity. And while in the case of Euclidean spaces and the usual topology open sets do use distance in a subtle way (since open sets depend on open balls, which use radius, which depends on distance), in the general topologies, where open sets are defined abstractly, without any distance whatsoever, continuity of functions is independent of that. So the right hand side of the lemma is exactly the right mathematical description of continuity, if we want to be able to stretch, compress, twist, deform "shapes" freely.

3 Homeomorphism and topological equivalence

Now that we have defined what a topological space (X, τ) is and we know when a set map $f : X \to Y$ is continuous with respect to some given topologies τ_X on X and τ_Y on Y is, we are ready to precisely formulate when two topological spaces are the same:

Definition 14 *The topological space* (X, τ_X) *and* (Y, τ_Y) *are topologically equivalent or homeomorphic, if there is a continuous bijection* $f : X \to Y$ *whose inverse* $f^{-1} : Y \to X$ *is also continuous.*

Exercise 15 *Verify that homeomorphism is an equivalence relation.*