0.1 Making new topologies from old

We will discuss three ways to obtain topologies from given one(s), using known set theoretical constructions. These are the subspace, the quotient and the product topologies.

0.1.1 The subspace topology

Suppose you have a set X with a topology τ_X on it. Let $A \subset X$ be arbitrary. One can obtain a topology on A using τ_X the following way.

Let

$$\tau_A = \{U \subset A \,|\, \exists V \in \tau_X \text{ s.t. } U = V \cap A\}$$

.

Exercise 0.1.1: Verify that this is a topology indeed.

Terminology: τ_A is called a subspace topology (with respect to τ_X).

Examples:

1. Let $X = \mathbb{R}$ with the usual topology. Consider $A = [0,1]$ with the subspace topology. Then $[0,1/2)$ is open in A, (but (still) not open in $X = \mathbb{R}$)

2. Let $X = \mathbb{R}$ with the usual topology. Consider $A = [0,1] \cup [3,4]$. Then $[0,1]$ and $[3,4]$ are open in A, but not open in $X = \mathbb{R}$.

3. Let $X = [0,1] \times [0,1]$ with the subspace topology determined by $(\mathbb{R}^2, \tau_{usual})$. Then the "half-open disk" $B_{(0,1/2)}(1/4) \cap X$ is open in X.

   (Here $B_{(0,1/2)}(1/4) = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + (y-1/2)^2 < 1/16\}$.)

Exercise 0.1.2: What is τ_A if $X = \mathbb{R}$ with the standard topology and $A = \mathbb{Z}$?

The following lemma is very useful and shows that in case of metric topologies, you can think of their subspaces as being generated by the restriction of the metric to the subset.

Lemma 0.1.3: Given a metric space (X,d), let τ_d be the topology generated by that metric. Consider also a subset $A \subset X$ and $d' = d_A$ the metric d restricted to this subset. Then

$$\tau_{d,A}, \text{ the subspace topology of } \tau_d \text{ on } A = \tau_{d'}, \text{ the metric topology on } A \text{ generated by } d' = d_A$$

0.1.2 The inclusion map.

Considering a set X and subset $A \subset X$ "comes with" a natural map - the inclusion map $i : A \to X$ where $i(a) = a \; \forall a \in A$.

From the definition of i we have $\forall V \subset X$ it follows that $i^{-1}(V) = V \cap A$.

Thus when X has topology τ_X and A is equipped with the corresponding subspace topology then i is automatically continuous.

Exercise 0.1.4: Verify this.

We also have the following technical lemma, which will be useful later.

Lemma 0.1.5: Given (X,τ_X), (Y,τ_Y) and $A \subset X$ with the subspace topology τ_A,
$f : Y \to A$ is continuous if and only if $i \circ f : Y \to X$ is continuous.

0.1.3 The quotient topology and the quotient map

First, let us recall the following "quotient set" construction from set theory.

Given a set X and an equivalence relation $\sim$ on it, the equivalence relation determines equivalence classes $\{A_\alpha\}$, where each $A_\alpha \subset X$, $A_\alpha \neq \emptyset$, their union $\cup A_\alpha = X$ and $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$.

The equivalence classes are defined by $a, b \in A_\alpha$ if and only if $a \sim b$.

The set of equivalence classes is called the quotient set. It is denoted by $X/\sim$ so that $X/\sim = \{A_\alpha\}$. So the elements of $X/\sim$ are subsets (equivalence classes) of X.

There is a "natural map" $q : X \to X/\sim$ coming with this setup, where $q(x) = [x] = A_\alpha =$ the equivalence class of x. (That is $q(x) = A_\alpha$ if $x \in A_\alpha$.)

The map q is called the quotient map. It is clearly onto.

Now, suppose that the set X also has a topology τ_X on it.

Then we can define a topology on $X/\sim$ by

$$\tau_{X/\sim} = \{U \subset X/\sim \,|\, q^{-1}(U) \in \tau_X\}$$

Exercise 0.1.6: Verify that this is a topology indeed.

Exercise 0.1.7: Think through that q is automatically continuous.

The following "technical lemma" will be useful later.

Lemma 0.1.8: Suppose we are given a topological space (X, τ_X), an equivalence relation $\sim$ on X, as well as a map $f : X/\sim \longrightarrow Y$ where Y is equipped with some topology τ_Y.

Then f is continuous if and only if $f \circ q : X \longrightarrow Y$ is continuous.

(Here $q : X \to X/\sim$ is the quotient map.)

Proof: $\Rightarrow$ Follows immediately, since the composition of continuous maps is continuous.

$\Leftarrow$. We have to show that if $U \subset Y$ is open in Y then the pre-image $f^{-1}(U)$ is open in $X/\sim$.

This is true, since by definition of the quotient topology, $f^{-1}(U)$ is open in $X/\sim$ happens exactly when the pre-image $q^{-1}(f^{-1}(U))$ is open in X. But $q^{-1}(f^{-1}(U)) = (f \circ q)^{-1}(U)$ and $f \circ q$ is continuous by assumption, so we are done.

Remark 0.1.9: Fix a set X. As we discussed above, each equivalence relation $\sim$ on X determines a partition P of X, namely, its partition into the equivalence classes determined by $\sim$.

Conversely, a given partition P of X – that is $\mathsf{P} = \{\mathsf{B}_\gamma\}$ where $B_\gamma \neq \emptyset$, $B_\gamma \subset X$, $B_\gamma \cap B_\delta = \emptyset$, if $\gamma \neq \delta$ and $\cup B_\gamma = X$ – determines an equivalence relation on X, by setting $a \sim b$ for each $a, b \in X$ if and only if $\exists B_\gamma \in \mathsf{P}$ such that $a, b \in B_\gamma$.

Exercise 0.1.10: Verify that "$a \sim b$ for each $a, b \in X$ if and only if $\exists B_\gamma \in \mathsf{P}$ such that $a, b \in B_\gamma$" given above is indeed an equivalence relation.

Thus every equivalence relation determines a partition and vice-versa.

Using this observation, one can construct quotient spaces using arbitrary partitions of the sets X in topological spaces $(X\tau_X)$.

Terminology: A quotient space is also referred to as an "identification space" with the equivalence relation determining it being the identification. When for points x, y of X we consider "$x \sim y$" we also say that "x is identified with y" (in $X/\sim$) or "x is glued to y" or "pasted to".

Example 0.1.11:

1. Let $X = [0,1]$ and consider the partition of X into $U = \{0,1\}$ as well as $V_a = \{a\}$ $\forall a \in (0,1)$. That is, consider $\mathsf{P} = \{\mathsf{U}, \mathsf{V_a} \,|\mathsf{a} \in (0,1)\}$.

   Equivalently, one can consider $\sim$ on X where $0 \sim 1$ and $a \sim b$ implies $a = b$ if $0 < a < b < 1$.

   Guess a well-known space that $X/\mathsf{P} = \mathsf{X}/\sim$ is homeomorphic to.

2. Let $X = \overline{D}^2$ = the closed unit disk centered at the origin in $\mathbb{R}^2$, as before (with the subspace topology of the standard topology on $\mathbb{R}^2$. Consider also the partition of X to $U = \{(x, y) \,|\, x^2 + y^2 = 1\}$ and all other points are in one-point sets.

   Equivalently, one can consider $\sim$ on X where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$ (and all other points are only eqivalent to themselves.

   Guess a well-known space that $X/\mathsf{P} = \mathsf{X}/\sim$ is homeomorphic to.

3. Let $X = [0,1] \times [0,1]$ (with the subspace topology of the standard topology of $\mathbb{R}^2$).

   Consider the identification $(0, y) \sim (1, y)$ for $y \in [0,1]$ (and all other points are identified with only themselves).

   Guess a well-known space that $X/\mathsf{P} = \mathsf{X}/\sim$ is homeomorphic to.

4. Let $X = [0,1] \times [0,1]$ (with the subspace topology of the standard topology of $\mathbb{R}^2$).

   Consider the identification $(0, y) \sim (1, 1-y)$ for $y \in [0,1]$ (and all other points are identified with only themselves).

   Definition 0.1.12: The quotient or identification space of this example is called *(open) Mobius strip*. It is denoted by M.

5. Let $X = [0,1] \times [0,1]$ (with the subspace topology of the standard topology of $\mathbb{R}^2$).

   Consider the identification $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$ for $x, y \in [0,1]$ (and all other points are identified with only themselves).

   Definition 0.1.13: The quotient or identification space of this example is called *torus*. It is denoted by T^2 or T.

6. Let $X = [0,1] \times [0,1]$ (with the subspace topology of the standard topology of $\mathbb{R}^2$).

   Consider the identification $(0, y) \sim (1, 1-y)$ and $(x, 0) \sim (x, 1)$ for $x, y \in [0,1]$ (and all other points are identified with only themselves).

Definition 0.1.14: The quotient or identification space of this example is called *Klein bottle*. It is denoted by K^2.

7. Let $X = \overline{D}^2 = \{(x,y) \,|\, x^2 + y^2 \leq 1\}$ (with the subspace topology of the standard topology of $\mathbb{R}^2$).

   Consider the identification $(x,y) \sim (-x,-y)$ when $x^2 + y^2 = 1$ (and all other points are identified with only themselves).

   Definition 0.1.15: The quotient or identification space of this example is called *real projective space*. It is denoted by $\mathbb{R}P^2$.

Examples 5-7 are all examples of *surfaces*.

Definition 0.1.16: A subset $S \subset \mathbb{R}^K$ ($K \in \mathbb{N}$) is a surface if $\forall p \in S$ $\exists U$ open set with $p \in U$ and $U \sim \mathbb{R}^2 \sim D^2$

(Above S has the subspace topology of the standard (or ususal) topology of $\mathbb{R}^K$.)

Thus every point of a surface "looks" locally just like the real Euclidean plane or, equivalently (up to homeomorphism) like an open disc in $\mathbb{R}^2$.

Example 0.1.17: More examples of surfaces include: $\mathbb{R}^2$ itself, the open disk D^2, the cylinder $C = \{(x,y,z) \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^3$, S^2.

Exercise 0.1.18: Check that the quotient construction is such that for each of the torus, Klein bottle and $\mathbb{R}P^2$ every point has an open set around it that is homeomorphic to an open disc of $\mathbb{R}^2$.

Above there are several examples that involve identifying edges of a square in $\mathbb{R}^2$ and as a result we got surfaces.

A natural question is then: if you take a square and consider some other identification of its edges, what do you get? Do you get something new or perhaps two such identifications provide the same quotient space?

More generally, if you take some a polygon and identify its edges, what do you get? How does one compare two such quotients? Especially, if the polygons we start with are different (have different number of sides)?

In order to be able to takle such and similar questions there is a special type of argument that topologists use - these are called *cut-and-paste arguments*.

Cut-and-paste arguments have been developed in variuos settings and higher dimensions as well. The version we consider is for identification spaces (or quotients) of polygons only.

0.1.4 Special quotients: identifying edges of polygons

Among the examples of quotient spaces we considered there were some some that involved identifying edges of a square in $\mathbb{R}^2$, pairwise. The torus and the Klein bottle were defined that way.

We also considered an example involving a 2-gon, that you can take to be the closed unit disk centered at the origin $\overline{D}^2 = \{(x,y)\,|\,x^2+y^2=1\}$ with the two edges being the upper and the lower unit semi-circles. We discussed that the identification $(x,y)\sim(-x,-y)$ when $x^2+y^2=1$ results in the quotient $\overline{D}^2/\sim$ which is the real projective plane $\mathbb{R}P^2$, by definition.
Alternatively, when $(x,y)\sim(x,-y)$ when $x^2+y^2=1$ we have $\overline{D}^2/\sim$ homeomorphic to S^2.

More generally, one can take any $2n$-gon $\mathsf{K}\subset\mathbb{R}^2$ $(n\geq 1)$ and identify its edges pairwise. The resulting quotient spaces all belong to a very important class of topological spaces, the *compact surfaces*, as we will see later.

Here is a precise description of this type of identification.

First of all, any edge can be oriented two ways: if the line segment PQ is an edge of a polygon K, where $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ are the vertices of K connected by the edge PQ, then the parametrization $\mathbf{r}(t)=(1-t)(x_1,y_1)+t(x_2,y_2)$ for $t\in[0,1]$ starts at P "runs along" PQ and ends at Q. This parametrization gives a direction to the edge and is denoted by putting an arrow on it in the direction pointing from P to Q.

The parametrization $\mathbf{r}(t)=t(x_1,y_1)+(1-t)(x_2,y_2)$ for $t\in[0,1]$ on the other hand starts at Q "runs along" the edge PQ ending at P. That is, it gives an orientation to the edge PQ that is opposite of the previous one.

This parametrization is denoted by putting an arrow on the edge in the direction from Q to P.

Now, what does identification of two edges of the polygon K mean?

Suppose the edge PQ between vertices $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is identified with another edge RS of the polygon K where $R=(x_3,y_3)$ and $S=(x_4,y_4)$. This can be done two different ways: without loss of generality assume that for $t\in[0,1]$ the edge PQ is parametrized by $\mathbf{r}(t)=(1-t)(x_1,y_1)+t(x_2,y_2)$ and assume that RS is parametrized by $\mathbf{r}'(t)=(1-t)(x_3,y_3)+t(x_4,y_4)$. Then setting $\mathbf{r}(t)\sim\mathbf{r}'(t)$ for $t\in[0,1]$ identifies (or "glues") the points of edge PQ to points of the edge RS.

Clearly, considering the opposite orientation on RS, that is the parametrization $\mathbf{r}'(t)=t(x_3,y_3)+(1-t)(x_4,y_4)$ for $t\in[0,1]$ and then again identifying according to $\mathbf{r}(t)\sim\mathbf{r}'(t)$ for $t\in[0,1]$ provides a different gluing of the edges PQ and RS, pointwise.

For example, in the first case P is identified with R and Q with S, while in the second case P is identifies with S and Q with R.

Diagram notation: When edges of a polygon are identified in pairs, the resulting quotient space is represented by that polygon where two edges are denoted by the same letter if and only if the two edges are identified. Arrows on the edges indicate how those two edges are identified *pointwise*, as described precisely above.

Words corresponding to diagrams: Each polygonal diagram as above determines a "word" on the letters of edge-labels $\{a_i\}$ as well as their formal inverses $\{a_i^{-1}\}$ the following way:

choose a vertex and a direction e.g. clockwise. Trace the edges of the polygon in clockwise direction, starting at your vertex – as you trace, write down the label of an edge with exponent +1 (usually omitted) if the direction of that edge coincides with the tracing-direction around the edges of the polygon (which was chosen to be clockwise), and write down the label of an edge with exponent -1 if the direction of that edge is opposite the "global" tracing-direction.

Remark 0.1.19: Note that if all edges of a polygon are labelled e.g. if you have a $2n$-gon whose edges are identified pairwise then the corresponding word is determined up to cyclic permutation.

Conversely, any word ω, where each letter a_i appears exactly twice, with exponents ± 1 determines a polygonal diagram, whose edges are identified in pairs.

The following are then natural questions: if one takes some polygon and identifies its edges pairwise, what quotient does one get? If we take different edge-identifications, do we get a different quotient spaces? How can we compare two such quotients? Especially, if the polygons we start with are different (have different number of vertices and edges)?

In order to be able to tackle such and similar questions there is a special type of argument that topologists use - these are called *cut-and-paste arguments.*

Cut-and-paste arguments have been developed in various settings. The version we consider is for identification spaces (or quotients) of polygons only.

0.1.5 Cut-and-paste arguments for polygons.

In order to understand a quotient space $X/\sim$ it may be useful to find another "view" of it, for example a quotient space $Y/\sim'$ such that the two are homeomorphic and the latter is "well-known".

In order to find such $Y/\sim'$ one can use the following **elementary operations on diagrams** as they do not change the corresponding quotient space, up to homeomorphism:

1. *Pasting or gluing along an edge with the same label, in the given direction.*

2. *Cutting along a new edge and remembering* – this is the reverse of the previous step.

3. *Relabelling* all occurrences of some edge-label by a label that does not appear anywhere else.

In particular, edges appearing consecutively with the same sequence of labels and directions throughout X (and nowhere else) can be replaced by a single edge, label and direction (at all occurences).

4. *Switching the direction of an edge, if all directions of all edges with the same label are switched.*

5. *Flipping.* A diagram corresponding to $a_1 ... a_n$ can be exchanged for a diagram $a_n^{-1} ... a_1^{-1}$

6. *Cancelling* If the diagram uses a polygon with more than two edges, then edges aa^{-1} can be omitted.

Note: intuitively, one can think of this as actually performing the gluing of edges aa^{-1}, as they are next to each other.

7. *Uncancelling* This is the reverse of the previous operation.

Note that Operations 3,4 and 5 are straightforward, if one thinks through how edge directions denote pointwise identification of edges.

Example 0.1.20: (of elementary operation 1.)
Suppose you have a diagram corresponding to words $w_1 = abc^{-1}$ and $w_2 = b^{-1}ca$. That is $X =$ two disjoint triangles whose edges are identified according to w_1 and w_2.

Clearly, $w_1 = c^{-1}ab$ and $w_2 = b^{-1}ca$ determine the same quotient. (The cyclic permutation in w_1 corresponds to starting reading the labels at another vertex of the first triangle.)

Then the diagram corresponding to $c^{-1}abb^{-1}ca$, that is $c^{-1}aca$, is the result of "pasting along b". In particular, gluing the two triangles corresponding to w_1 and w_2 respectively, along the edge b, results in a square with labelling word $c^{-1}aca$.

Example 0.1.21: (of elementary operation 3.) If a diagram corresponds to the word $\omega = \omega_1\, fgh^{-1}\omega_2\, fgh^{-1}\omega_3\, hg^{-1}f^{-1}$ and the subwords $\omega_1, \omega_2, \omega_3$ do not contain labelling letters f, g, h then setting $s = fgh^{-1}$, the diagram corresponding to ω can be replaced by a diagram corresponding to $\omega_1\, s\omega_2\, s\omega_3\, s^{-1}$

Definition 0.1.22: Given polygons X and Y and some identifications of their edges which by abuse of notation will be denoted $\sim$ (although clearly may be very different identifications even if X and Y are the same), **the diagram corresponding to $X/\!\sim$ is equivalent to the diagram corresponding to $Y/\sim$** if there is a sequence of elementary operations 1-7 above, that transforms one to the other.

Proposition 0.1.23: If the diagrams corresponding to quotients of polygons X and Y with respect to identifying their edges are equivalent then the quotient spaces $X/\!\sim$ and $Y/\sim$ are homeomorphic.

Finally, "cut-and-paste arguments" on diagrams i.e. in case of identification spaces ("quotients") of polygons whose certain edges are identified in pairs, can for example be used to prove that two different quotient spaces are homeomorphic.

Or investigate what happens when a given quotient space is altered (e.g. by a cut etc).

Use cut-and-paste arguments to solve the following exercises:

Exercise 0.1.24: What do you get when you cut a Mobius strip along its "middle circle", parallel to its edge?
That is, if $M = [0,1]\times[0,1]/(0,y) \sim (1,1-y)$ then the "middle circle" is $A = \{(0,1/2)\}/[0,1]/(0,1/2) \sim (1,1/2)$.

Exercise 0.1.25: Show that when you glue a closed disk onto a Mobius strip, along its edge, you get a real projective plane.

Exercise 0.1.26: Show that two Mobius strips glued along their boundaries give a Klein bottle.

Exercise 0.1.27: Show that $abca^{-1}b^{-1}c^{-1}$ determines a torus. *($abca^{-1}b^{-1}c^{-1}$ is called the "hexagonal torus".)*

0.1.6 The product topology and quotient maps

The following is well-know from set theory. Given sets X and Y, their Cartesian product is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

A very important property is that $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$. Thus for each $(x, y) \in X \times Y$ the subset $\{x\} \times Y$ is bijective to Y, $X \times \{y\}$ is bijective to X and

$$\{x\} \times Y \ \cap \ X \times \{y\} = \{(x, y)\}.$$

Given X, Y and $X \times Y$, the "natural" maps $pr_X : X \times Y \longrightarrow X$ defined by $(x, y) \mapsto x$ and $pr_Y : X \times Y \longrightarrow Y$ defined by $(x, y) \mapsto y$ are called the projection maps onto the X and Y factor, respectively.

Note that $\forall V \subset X$, we have for the pre-image $pr_X^{-1}(V) = V \times Y$ and similarly, $\forall W \subset Y$, we have for the pre-image $pr_Y^{-1}(W) = X \times W$.

Now, suppose that X has topology τ_X and Y has τ_Y. Using these topologies we define a topology on $X \times Y$ in two steps.

Step 1. Let

$$\mathcal{B} = \{U \times V \mid U \in \tau_X, \, V \in \tau_Y\}$$

Step 2. Let

$$\tau_{X \times Y} = \{\cup_\alpha B_\alpha \mid B_\alpha \in \mathcal{B}\}$$

Exercise 0.1.28: $\mathcal{B}$ is not a topology – why?. Verify that $\tau_{X \times Y}$ is a topology indeed.

Definition 0.1.29: $\tau_{X \times Y}$ is called the product topology on $X \times Y$ induced by τ_X and τ_Y.

Exercise 0.1.30: Show that W is open in $X \times Y$ is and only if $\forall p \in W$ there exist $U \in \tau_X$ and $V \in \tau_Y$ with $p \in U \times V \subset W$.

Example 0.1.31:

1. Let $X = Y = \mathbb{R}$ with the usual topology. The product topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the standard topology.

2. Let $C = \{(x, y, z) | x^2 + y^2 = 1\}$. Give an intuitive explanation why C can be thought of as $S^1 \times \mathbb{R}$.

3. The annulus $A = \{(r, \theta) \,|\, 1 \leq r \leq 2\}$ can be thought of as $[1, 2] \times S^1$

4. $S^1 \times S^1$ is homeomorphic to a torus. Give an intuitive explanation.

5. Let $X = [0, 1] \times [0, 1] \times [0, 1]$ (with the subspace topology of the standard topology of $\mathbb{R}^3$).

   Consider the identification $(0, y, z) \sim (1, y, z)$, $(x, 0, z) \sim (x, 1, z)$, $(x, y, 0) \sim (x, y, 1)$ (and all other points are identified with only themselves).

   Definition 0.1.32: The quotient (or identification) space of this example is called the *3-torus*. It is denoted by T^3.

This quotient space has a product structure $T^2 \times S^1$ (where T^2) is the (usual) torus. Give an intuitive explanation as to why. Note that since T^2 has a product structure of $S^1 \times S^1$, T^3 has $S^1 \times S^1 \times S^1$. Where can that "be seen" in the gluing" (ie quotient space of the cube)?

Exercise 0.1.33: pr_X and pr_Y are continuous.

Moreover, they provide homeomorphisms between X and $X \times \{y\}$ as well as Y and $\{x\} \times Y$ for each $x \in X$ and $y \in Y$.

Thus $X \times \{y\}$ and $\{x\} \times Y$ can be thought of as "copies" of X and Y at (x, y).

We have the following technical lemma.

Lemma 0.1.34: Given (X, τ_X), (Y, τ_Y) and (Z, τ_Z). Suppose $X \times Y$ is equipped with the product topology.

Then $f : Z \to X \times Y$ is continuous if and only if the compositions $pr_X \circ f : Z \to X$ and $pr_Y \circ f : Z \to Y$ are continuous.

Proof: $\Rightarrow$ is immediate, since the composition of continuous maps is continuous.

$\Leftarrow$. We have to show that for each $W \subset X \times Y$ that is open in $X \times Y$, the pre-image $f^{-1}(W) \in \tau_Z$.

We do this in two steps.

Step 1. Assume that $W \in \mathcal{B}$, that is $W = U \times V$ where $U \in \tau_X$ and $V \in \tau_Y$. Note that we have the following equivalence of sets:

$$U \times V = (U \times Y) \cap (X \times V).$$

Also $U \times Y = pr_X^{-1}(U)$ and $X \times V = pr_Y^{-1}(V)$. Thus

$$f^{-1}(U \times V) = f^{-1}((U \times Y) \cap (X \times V)) = f^{-1}(U \times Y) \cap f^{-1}(X \times V) =$$

$$= f^{-1}(pr_X^{-1}(U)) \cap f^{-1}(pr_Y^{-1}(V)) = (pr_X \circ f)^{-1}(U) \cap (pr_Y \circ f)^{-1}(V)$$

Since $pr_X \circ f$ and $pr_Y \circ f$ are continuous by assumption, the two sets in the last expressions are open in Z and thus their intersection is also open and we are done with step 1.

Now, for step 2, assume that W is an arbitrary open set in $X \times Y$. Then $W = \cup_\alpha B_\alpha$ where $B_\alpha in \mathcal{B}$. Then

$$f^{-1}(W) = f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha)$$

By step 1, each of the sets in the last union is open in Z, so the union is also open in Z and we are done.

Exercise 0.1.35: Finally, here are two advanced examples.

1. Consider $S^1 \times \overline{D}^2$, where S^1 is the unit circle and $\overline{D}^2$ is the (closed) unit disc in $\mathbb{R}^2$. The product $S^1 \times \overline{D}^2$ can be thought of as a solid torus (a doughnut filled with dough).

Now, take another copy of the solid torus by considering the original one and take a translation of it so that the original and its translation do not intersect. Glue the two solid tori by identifying corresponding points of their bounding surfaces (i.e. each original point by the translation of it).

We get a product space again. What is it? Give an intuitive explanation.

2. Start with the solid cube $X = \{(x, y, z) \,|\, x, z, y \in [-1, 1]\}$. Glue its faces as follows: $(1, y, z) \sim (-1, y, z)$, $(x, 1, z) \sim (-x, -1, z)$ and $(x, y, 1) \sim (x, y, -1)$. Guess an alternative product description of X.