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Referee's Report on the PhD thesis

First-Order Logic Investigation of Relativity Theory with an Emphasis on Accelerated Observers

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Summary

This thesis studies the logical foundations of a number of aspects of the geometry and physics of relativity theory, both special and general, by formulating them in axiomatic theories based on suitable first-order languages. As explained in the Introduction and in Chapter 11, the use of first-order logic has a number of advantages. It makes it possible to express all concepts in a precise way, eliminating tacit assumptions, and without any dependence on set-theoretic hypotheses. Moreover, first-order logic has a complete proof theory, which means that to show that a sentence is derivable from a certain set of sentences, it is enough to show that it is true in any model of the set of sentences. First-order logic also has well-developed techniques for building and analysing models. The use of the axiomatic method makes it possible to identify just what properties of a physical theory, expressed as axioms, are responsible for its consequences (e.g. the twin paradox). This combination of logic and geometry is in the spirit of Tarski's axiomatisation of Euclidean and other geometries, and ultimately is a continuation of a tradition that goes back to Euclid himself. The thesis under review was developed in the Budapest group on the logical foundations of Relativity that is led by Andréka and Németi, and is a powerful and sophisticated application of the logical method.

Chapters 2 and 3 review the formalism established by the Budapest group, describing the basic first-order language used and some fundamental concepts, which are then used to define the axiom systems $\mathsf{SpecRel}$ and $\mathsf{SpecRel}_0$ for Special Relativity.

Chapter 4 gives a formulation of the famous Clock Paradox (CP) and two variants of it (Anti-CP and No-CP), and then derives geometrical characterisations of these principles within models of a weak theory Kinem of kinematics that is a subtheory both of Newtonian kinematics and Special Relativity. The Newtonian assumption $\mathsf{AbsTime}$ of absolute time is shown to imply No-CP, while conversely, No-CP implies $\mathsf{AbsTime}$ under the additional assumption that inertial observers can move in any direction at any finite speed. But if this last assumption is weakened to require only that inertial observers can move in any direction at a speed which is arbitrarily close to any finite speed, then No-CP does *not* imply $\mathsf{AbsTime}$ over Kinem. That is shown by construction of a suitable counter-model. These results already give a very impressive illustration of the ability of the axiomatic method to clarify the assumptions underlying scientific principles. Similar results are obtained to show, over models of $\mathsf{SpecRel}_0$ together with other principles, that CP is logically weaker than the principle $\mathsf{SlowTime}$ of slowing down of relatively moving clocks, and weaker than a principle $\mathsf{AxSymDist}$ asserting that inertial observers agree on the spatial distance

between events that are simultaneous for both of them. It is also shown that SlowTime and AxSymDist are equivalent over these models if the quantity part is the set of real numbers – a condition that is not first-order expressible.

Chapter 5 makes a contribution to the logical analysis of relativistic dynamics, introducing axioms capturing properties of relativistic mass as measured by the effects of collisions. It shows the way in which certain geometrical axioms about the linear behaviour of the center of mass of colliding intertial bodies can replace standard physical axioms about the conservation of mass and momentum. In particular a geometrical theory is obtained within which the well-known relationship $m_0 = \sqrt{1 - v^2/c^2} \cdot m$ between rest mass and relativistic mass can be derived. It is pointed out this can then be used to derive the Einstein equation $E = mc^2$ in this theory.

Chapter 6 extends the axiomatisation of special relativity to accomodate *accelerated* observers. The main new axiom AxCmv asserts that at any event encountered by any observer there is a "co-moving" intertial observer who assigns the same coordinates to the nearby world for a short time. A logic AccRel_0 is defined that extends SpecRel by AccRel_0 and some simplifying axioms. A general construction is given for models of AccRel_0.

This system is then used in Chapter 7 to discuss a formulation TwP of the twin paradox, which is the accelerated version of the clock paradox. It is shown that if $\mathfrak{Q}$ is any Euclidean ordered field not isomorphic to the real number field $\mathbb{R}$, then there is a model of AccRel_0 whose quantity part is $\mathfrak{Q}$, such that this model does not validate TwP or a principle DDPE asserting that the clocks of two observers with the same world line are synchronized. Thus neither TwP nor DDPE are derivable in the logic $\mathsf{AccRel}_0 + Th(\mathbb{R})$, where $Th(\mathbb{R})$ is the first-order theory of the real number field. It is then shown that for any Euclidean ordered field $\mathfrak{Q}$ that is non-Archimedean or countable, there is a model of AccRel_0 whose quantity part is $\mathfrak{Q}$, such that this model does not validate TwP or DDPE, but satisfies a number of physically natural properties, including that all observers use the whole coordinate system, coordinatize the same events, and coordinatize every event only once.

The missing property needed to derive the twin paradox over AccRel_0 is identified as the axiom schema CONT asserting that every non-empty bounded subset of the quantity part that is definable by a first-order formula has a supremum. It is shown that for spacetime dimension at least 3, TwP and DDPE are both consequences of the theory $\mathsf{AccRel} = \mathsf{AccRel}_0 + \mathsf{CONT}$.

Chapter 8 uses the theory AccRel to study gravitational time dilation. Under Einstein's principle of the equivalence of gravitation and acceleration, this can be formulated in terms of time running more slowly at the back of an accelerated spaceship than in the front. A "spaceship" is taken to be a triple of coplanar observers b, k, c with k at constant distance from b and c, where distance is measured either by "radar" or by the Minkowskian metric. AccRel is shown to imply that, in essence, when the spaceship has the same direction as a positively acceleration k, the clock of b runs slower than that of c as observed by k when measured by radar or by photons or by the Minkowskian metric. It is also shown that there is a model of AccRel with two observers b, c with the clock of b running slower than c as observed by b (by any of the methods) while b and c have the same acceleration, or equivalently experience the same gravitation. This suggests that it is the direction rather than magnitude of gravitation that makes time slow down. Theorems are also given stating that clocks can run arbitrarily slow or fast according to the three different methods. These results require an axiom schema COMP expressing that any first-order definable timelike curve is the world-line of some observer.

Chapter 9 modifies the system AccRel to obtain an axiomatic theory for general relativity. The essential idea is to refine the axioms of AccRel_0 + AxCmv to eliminate reference to *inertial* observers, in such a way that the new axioms are consequences of the old ones together with

AccRel_0. The new axioms are combined with CONT and an axiom Diff_n asserting that world-view transformations are n-times differentiable, to obtain a system GenRel_n. Adding all the axioms Diff_n results in a system GenRel_ω. It is explained that GenRel_n is complete with respect to n-times differentiable Lorentzian manifolds over real closed fields, while GenRel_ω is is complete with respect to smooth Lorentzian manifolds over real closed fields.

Chapter 10 develops the concepts and results from real analysis that are needed to prove the main theorems of the thesis. This includes demonstrating that many standard results about continuous and differentiable functions over $\mathbb{R}$ can be shown to hold for definable functions over the first-order theory of ordered fields, in some cases invoking the definable-continuity schema CONT. These results are then applied to the required analysis of definable timelike curves.

Evaluation

This is a very high quality thesis that clearly merits the award of the PhD degree.

The candidate demonstrates that he has absorbed a great deal of literature concerned with logic and relativistic physics, and has acquired a deep understanding of it. He is able to analyse the relationships between key concepts, formulate questions about them, find solutions, and draw conclusions about their interpretation. The exposition is excellent: explanations are given in a clear, direct and readable style.

The thesis contains original results that are important to the field and make a real contribution to the advancement of knowledge, in fact a substantial body of contributions. Relativity theory is fundamental to modern physical science, but its concepts have always been challenging to comprehend. The thesis contributes significantly to this comprehension through its logical investigation of the principles underlying the clock and twin paradoxes, the analysis of relativistic dynamics, and the complete axiomatisation of the theory for general relativity over Lorentzian manifolds.

A highlight is the result, concerning the twin paradox, that TwP is derivable in $\mathsf{AccRel}_0 + \mathsf{CONT}$ but not in $\mathsf{AccRel}_0 + Th(\mathbb{R})$. This shows that TwP depends on definable continuity, not just of formulas from the first-order theory of real-closed fields, but also of formulas involving the primitive relativistic notions (world-view relation, photons, observers etc.). I was also impressed by the material in Chapter 10 developing first-order real analysis. This is of interest in its own right, and the treatment displays considerable mathematical sophistication, insight, and problem-solving ability.

Quality

I judge the thesis to be in the category *summa cum laude.*

Minor editorial points

page 57, line 5: "it is a $\vec{q}$" should be "it is $\vec{q}$"

page 71, middle column, lower diagram: "forth" should be "fourth" page 95, 2nd row of formula AxPh^{-}: should $\vec{v}^{ph}_{k}(\vec{p}) = \vec{v}$ be $\vec{\mathbf{v}}^{ph}_{k}(\vec{p}) = \vec{v}$?

page 97, line 3: “if its every point” would be better expressed “if each of its points”

page 98, 2nd-to-last line: better to say “$\langle \mathrm{Q}, \leq \rangle$ is a partially ordered set”.

pages 102, 103: Bolzano’s Theorem is also commonly known as the Intermediate Value Theorem.

page 104, line 4 *and many other places*: the word “monotonous” has negative connotations in English. My dictionary defines it as meaning “dull, tedious and repetitious; lacking in variety and interest”. The mathematical concept of order-preserving here is usually called “monotonic”.

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