

# A Geometrical Characterization of the Twin Paradox and its Variants

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**Abstract.** The aim of this paper is to provide a logic-based conceptual analysis of the twin paradox (TwP) theorem within a first-order logic framework. A geometrical characterization of TwP and its variants is given. It is shown that TwP is not logically equivalent to the assumption of the slowing down of moving clocks, and the lack of TwP is not logically equivalent to the Newtonian assumption of absolute time. The logical connection between TwP and a symmetry axiom of special relativity is also studied.

*Keywords:* twin paradox; geometrical characterization; logical foundations; axiomatization; special relativity

## 1. Introduction

The twin paradox (TwP) theorem is one of the most famous predictions of special relativity. According to TwP, if a twin makes a journey into space, he will return to find that he has aged less than his twin brother who stayed at home. However surprising TwP is, it is not a contradiction. It is only a fact that shows that the concept of time is not as simple as it seems to be.\*

A more optimistic consequence of TwP is the following. Suppose you would like to visit a distant galaxy 200 light years away. You are told it is impossible because even light travels there for 200 years. But you do not

\*Unfortunately, it is still not uncommon for people who misinterpret the word ‘paradox’ to try to find contradictions in relativity theory, that is why we think it important to note here that its original meaning is “a statement that is seemingly contradictory and yet is actually true,” that is, it has nothing to do with logical contradiction. With the nearly century long fruitless debate in view, perhaps it would be better to call the paradoxes of relativity theory simply effects, thus saying “twin effect” instead of “twin paradox,” but for the time being it appears to be a hopeless effort to have this idea generally accepted. Anyway, we would like to emphasize that it is absolutely pointless to try to find a logical contradiction in relativity theory, as its consistency has been proved, see [3, Corollary 11.12, p.644], [1, p.77].

despair, you accelerate your spaceship nearly to the speed of light. Then you travel there in 1 year of your (proper) time. You study there whatever you wanted, and you come back in 1 year. When you arrive back, you aged only 2 years. So you are happy, but of course you cannot tell the story to your brother, who stayed on Earth. Alas you can tell it to your grand-...-grand-children only. In this way TwP also makes time travel to the future possible.

In this paper we use the axiomatic method to provide a logic-based conceptual analysis of the TwP theorem. We work within the first-order logic (FOL) framework of [1], [2], [3]. We logically compare TwP and a prediction (slowing down of moving clocks) as well as a symmetry axiom of special relativity. This analysis is based on our geometrical characterization of TwP, see Theorem 1. We show that TwP is logically weaker than the assumption of the slowing down of moving clocks, see Theorem 5. We also show that TwP is logically weaker than a symmetry axiom of special relativity, see Theorem 6. Since we prove our geometrical characterization in a general kinematics setting, we can use it to derive consequences on Newtonian kinematics too. We show that the absoluteness of time (in the Newtonian sense) is not equivalent to the lack of the twin paradox (No-TwP) without assuming a strong theoretical axiom, see Theorem 2.

Why is it useful to apply the axiomatic method to relativity theory? For one thing, this method makes it possible to understand the role of any particular axiom. We can check what happens to our theory if we drop, weaken or replace an axiom. For instance, it has been shown by this method that the impossibility of faster than light motion is not independent from other assumptions of special relativity, see [1, §3.4], [2]. More boldly: it is superfluous as an axiom because it is provable as a theorem from much simpler and more convincing basic assumptions. The linearity of transformations between inertial observers (inertial reference frames) can also be proven from some plausible assumptions, therefore it need not be assumed as an axiom, see [1], [2].

The usual approaches to special theory of relativity base the theory on two postulates, namely, Einstein's principle of relativity and that the velocity of light is independent of its source. Some authors give a mathematical argument to prove that Einstein's principle of relativity implies the second postulate, see, e.g., [9], [23]. However, these approaches contain several tacit assumptions besides the named postulates. So from the point of view of axiomatic foundations of relativity theory, they are not explicit enough. In an adequate axiomatic foundational work it is desirable to state every

assumption explicitly.<sup>†</sup>

Getting rid of unnecessary axioms of a physical theory is important because we do not know whether an axiom is true or not, we just assume so. We can only be sure of outcomes of concrete experiments but they rather correspond to (existentially quantified) theorems and not to axioms. In the literature it is common to use the term “empirical fact” for universal generalization of an empirical fact (elevated to the level of axioms), see, e.g., [11, §4], [26]. However, because of their falsifiability it would be better to call them empirical axioms (postulates based on outcomes of concrete experiments).

Similarly, if we axiomatize a theory, we can ask which axioms are responsible for a certain prediction of the theory. This kind of reverse thinking helps to answer the why-type questions of relativity. For example, we can take the twin paradox and check which axiom of special relativity was and which one was not needed to derive it. The weaker an axiom system is, the better answer it offers to the question: “Why is the twin paradox true?”. For details on answering why-type questions of relativity by the methodology of the present work, see [30]. For further reasons why to apply the axiomatic method to spacetime theories, see, e.g., [1], [2], [12], [20], [24].

Applying mathematical logic in foundations of relativity theories is not a new idea at all. It goes back to such leading mathematicians and philosophers as Hilbert, Reichenbach, Carnap, Gödel, Tarski, Suppes and Friedman, among others. The work of our school of Logic and Relativity led by Andréka and Némethi is continuation to their research. In a spirit similar to ours, there is a large variety of works devoted to logical axiomatizations of relativity, see, e.g., Ax [4], Benda [5], Goldblatt [10], Mundy [16], [17], Pambuccian [18], Robb [19], Suppes [25], Schutz [20], [21], [22].

Our general aims are to axiomatize relativity theories within pure FOL using simple, comprehensible and transparent basic assumptions (axioms) only; to prove the surprising predictions (theorems) of relativity theories from a minimal number of convincing axioms; to eliminate tacit assumptions from relativity by replacing them with explicit axioms formulated in FOL (in the spirit of the FOL foundation of mathematics and Tarski’s axiomatization of geometry); and to provide a foundation for relativity theory similar to that of mathematics, cf. Hilbert’s 6th problem [7]. In our perspective axiomatization is only a first step to logical and conceptual analysis where the real fun begins.

<sup>†</sup>The logical formulation of Einstein’s principle of relativity is not an easy task since it is difficult to capture axiomatically what “the laws of nature” are. Therefore we will use a different approach here. For details on the axiomatic reformulation of Einstein’s principle of relativity, see [1], [13, §2.8.3].

For good reasons, the foundation of mathematics was performed strictly within FOL. A reason for this fact is that staying within FOL helps to avoid tacit assumptions. Another reason is that FOL has a complete inference system while second-order logic (and thus any higher-order logic) cannot have one. For further reasons why to stay within FOL when dealing with axiomatic foundations, see, e.g., [1, §Appendix: Why FOL?], [4], [31], [34].

## 2. A FOL axiom system of kinematics

Here we explain our basic concepts. We deal with kinematics, i.e., with the motion of *bodies* (anything which can move, e.g., test-particles, reference frames, electromagnetic waves or centers of mass). We represent motion as the changing of spatial location in time. Thus we use reference frames for coordinatizing events (meetings of bodies). *Quantities* are used for marking time and space. The structure of quantities is assumed to be an *ordered field* in place of the field of real numbers.<sup>‡</sup> For simplicity, we associate reference frames with special bodies which we call *observers*.<sup>§</sup> Observations are formulated by means of the *worldview relation*.

There are several reasons for using observers (or coordinate systems, or reference frames) instead of a single observer-independent spacetime structure. One is that it helps to weed out unnecessary axioms from our theories. Nevertheless, we state and emphasize the logical equivalence<sup>¶</sup> of observer-oriented and observer-independent approaches to relativity theory, see, e.g., [13, §4.5].

Keeping the foregoing in mind, let us now set up the FOL language of our axiom systems. First we fix a natural number  $d \geq 2$  for the dimension of spacetime. We use a two-sorted language: **B** is the sort of (potential) **bodies** and **Q** is the sort of **quantities**. Our language contains the following non-logical symbols: unary relation symbol **IOb** (inertial **observers**); binary function symbols **+**, **·** and a binary relation symbol **<** (the field operations and the ordering on **Q**); and a  $2 + d$ -ary relation symbol **W** (**worldview relation**).

<sup>‡</sup>Using ordered fields in place of the field of real numbers increases the flexibility of the theory and minimizes the amount of mathematical presuppositions. For further motivation in this direction, see, e.g., Ax [4]. Similar remarks apply to our other flexibility-oriented decisions, e.g., to keep the dimension of spacetime as a variable.

<sup>§</sup>The body associated to a reference frame is nothing else than a label on the reference frame making it easier to talk about its motion.

<sup>¶</sup>By logical equivalence, we mean definitional equivalence.

The variables of sort  $\mathbf{B}$  are denoted by  $m, k, a, b$  and  $c$ ; and those of sort  $\mathbf{Q}$  are denoted by  $p, q, r, x$  and  $y$ .  $\text{IOb}(m)$  is translated as “ $m$  is an (inertial) observer.” We use the worldview relation  $W$  to speak about coordinatization by translating  $W(m, b, x_1, \dots, x_d)$  as “observer  $m$  coordinatizes body  $b$  at spacetime location  $\langle x_1, \dots, x_d \rangle$ ,” that is, at space location  $\langle x_2, \dots, x_d \rangle$  at instant  $x_1$ .

Body terms are just the variables of sort  $\mathbf{B}$ . Quantity terms are the variables of sort  $\mathbf{Q}$  and what can be built up from quantity terms by using the field operations  $(+, \cdot)$ .  $\text{IOb}(m)$ ,  $W(m, b, x_1, \dots, x_d)$ ,  $m = b$ ,  $x_1 = x_2$  and  $x_1 < x_2$  are the so-called atomic formulas of our FOL language, where  $m, b, x_1, \dots, x_d$  can be arbitrary terms of the required sorts. The **formulas** of our FOL language are built up from these atomic formulas by using the logical connectives *not* ( $\neg$ ), *and* ( $\wedge$ ), *or* ( $\vee$ ), *implies* ( $\rightarrow$ ), *if-and-only-if* ( $\leftrightarrow$ ) and the quantifiers *exists*  $x$  ( $\exists x$ ) and *for all*  $x$  ( $\forall x$ ) for every variable  $x$ . To abbreviate formulas of FOL we often omit parentheses according to the following convention. Quantifiers bind as long as they can, and  $\wedge$  binds stronger than  $\rightarrow$ . For example,  $\forall x \varphi \wedge \psi \rightarrow \exists y \delta \wedge \eta$  means  $\forall x ((\varphi \wedge \psi) \rightarrow \exists y (\delta \wedge \eta))$ .

We use first-order set theory as a meta theory to speak about model theoretical terms, such as models, validity, etc. The **models** of this language are of the form

$$\mathfrak{M} = \langle \mathbf{B}, \mathbf{Q}; \text{IOb}_{\mathfrak{M}}, +_{\mathfrak{M}}, \cdot_{\mathfrak{M}}, <_{\mathfrak{M}}, W_{\mathfrak{M}} \rangle,$$

where  $\mathbf{B}$  and  $\mathbf{Q}$  are nonempty sets and  $\text{IOb}_{\mathfrak{M}}$  is a unary relation on  $\mathbf{B}$ ,  $+_{\mathfrak{M}}$  and  $\cdot_{\mathfrak{M}}$  are binary functions and  $<_{\mathfrak{M}}$  is a binary relation on  $\mathbf{Q}$ , and  $W_{\mathfrak{M}}$  is a relation on  $\mathbf{B} \times \mathbf{B} \times \mathbf{Q} \times \dots \times \mathbf{Q}$ . Formulas are interpreted in  $\mathfrak{M}$  in the usual way.

We formulate each axiom at two levels. First we give an intuitive formulation, then a precise formalization using our logical notation (which can easily be translated into FOL formulas by inserting the FOL definitions into the formalizations). We seek to formulate easily understandable axioms in FOL.

We use the notation  $\mathbf{Q}^n := \mathbf{Q} \times \dots \times \mathbf{Q}$  ( $n$ -times) for the set of all  $n$ -tuples of elements of  $\mathbf{Q}$ . If  $p \in \mathbf{Q}^n$ , we assume that  $p = \langle p_1, \dots, p_n \rangle$ , i.e.,  $p_i \in \mathbf{Q}$  denotes the  $i$ -th component of the  $n$ -tuple  $p$ . Specially, we write  $W(m, b, p)$  in place of  $W(m, b, p_1, \dots, p_d)$ , and we write  $\forall p$  in place of  $\forall p_1 \dots \forall p_d$ , etc. To abbreviate formulas, we also use bounded quantifiers in the following way:  $\forall x \in \varphi \psi$  and  $\exists x \in \varphi \psi$  are abbreviated to  $\forall x \in \varphi \psi$  and  $\exists x \in \varphi \psi$ ,

respectively. For example, we write

$$\forall m \in \text{IOb} \exists b \in \text{B} \exists p \in \mathbb{Q}^d \quad W(m, b, p)$$

instead of

$$\forall m \text{IOb}(m) \rightarrow \exists b \text{B}(b) \wedge \exists p \text{Q}(p_1) \wedge \dots \wedge \text{Q}(p_d) \wedge W(m, b, p)$$

to formulate that every observer observes a body somewhere.

To be able to add, multiply and compare measurements by observers, we provide an algebraic structure for the set of quantities by our first axiom.

**AxEOF** The **quantity part**  $\langle \mathbb{Q}; +, \cdot, < \rangle$  is a Euclidean ordered field (i.e., a linearly ordered field in which positive elements have square roots).

For the FOL definition of linearly ordered field, see, e.g., [6]. We use the usual field operations  $0, 1, -, /, \sqrt{\quad}$  definable within FOL. We also use the vector-space structure of  $\mathbb{Q}^n$ , i.e., if  $p, q \in \mathbb{Q}^n$  and  $\lambda \in \mathbb{Q}$ , then  $p + q, -p, \lambda \cdot p \in \mathbb{Q}^n$ ; the **length** of  $p \in \mathbb{Q}^n$  is defined as

$$|p| := \sqrt{p_1^2 + \dots + p_n^2}$$

for any  $n \geq 1$ , and  $o := \langle 0, \dots, 0 \rangle$  denotes the **origin**. The set of positive elements of  $\mathbb{Q}$  (i.e., the set  $\{x \in \mathbb{Q} : 0 < x\}$ ) is denoted by  $\mathbb{Q}^+$ .

We need some definitions and notations to formulate our other axioms. The set  $\mathbb{Q}^d$  is called the **coordinate system** and its elements are referred to as **coordinate points**. We use the notations

$$p_\sigma := \langle p_2, \dots, p_d \rangle \quad \text{and} \quad p_\tau := p_1$$

for the **space component** and the **time component** of  $p \in \mathbb{Q}^d$ , respectively.

Our first axiom on observers simply states that each observer thinks that it is stationary in the origin of the space part of its coordinate system.

**AxSelf** An observer observes itself at a coordinate point iff the space component of this point is the origin:

$$\forall m \in \text{IOb} \forall p \in \mathbb{Q}^d \quad W(m, m, p) \leftrightarrow p_\sigma = o.$$

The **event** (the set of bodies) observed by observer  $m$  at coordinate point  $p$  is denoted by  $\text{ev}_m(p)$ , i.e.,

$$\text{ev}_m(p) := \{ b \in \text{B} : W(m, b, p) \},$$



where  $R^{-1} := \{\langle y, x \rangle : \langle x, y \rangle \in R\}$  is the FOL definition of the **inverse** of binary relation  $R$ . Let us note that by this definition, coordinate-function  $Crd_m$  may not be a function (in case  $ev_m$  is not one-to-one). It is only a binary relation.

**Convention 1.** Whenever we write  $Crd_m(e)$ , we mean that there is a unique  $q \in \mathbb{Q}^d$  such that  $ev_m(q) = e$ , and this  $q$  is denoted by  $Crd_m(e)$ . That is, if we talk about the value  $Crd_m(e)$ , we postulate that it exists and is unique.

The **time of event**  $e$  according to observer  $m$  is defined as

$$\mathbf{time}_m(e) := Crd_m(e)_\tau,$$

and the **elapsed time** between events  $e_1$  and  $e_2$  measured by observer  $m$  is defined as

$$\mathbf{time}_m(e_1, e_2) := |\mathbf{time}_m(e_1) - \mathbf{time}_m(e_2)|;$$

$\mathbf{time}_m(e_1, e_2)$  is called the *proper time* measured by  $m$  between  $e_1$  and  $e_2$  if  $m \in e_1 \cap e_2$ . Let us note that whenever we write  $\mathbf{time}_m$ , we assume that the events in its argument have unique coordinates by Convention 1.

The **coordinate-domain** of observer  $m$ , in symbols  $Cd_m$ , is the set of coordinate points where  $m$  observes something, i.e.,

$$Cd_m := \{p \in \mathbb{Q}^d : ev_m(p) \neq \emptyset\}.$$

The **worldview transformation** between the coordinate-domains of observers  $k$  and  $m$  is defined as

$$w_m^k := \{\langle q, p \rangle \in Cd_k \times Cd_m : ev_k(q) = ev_m(p)\}.$$

Let us note that worldview transformations are only binary relations by this definition.

**Convention 2.** Whenever we write  $w_m^k(q)$ , we mean there is a unique  $p \in \mathbb{Q}^d$  such that  $\langle q, p \rangle \in w_m^k$ , and this  $p$  is denoted by  $w_m^k(q)$ .

Let  $\mathbf{1}_t := \langle 1, 0, \dots, 0 \rangle$ . The **time-unit vector** of  $k$  according to  $m$  is defined as

$$\mathbf{1}_m^k := w_m^k(\mathbf{1}_t) - w_m^k(o).$$

The **world-line** of body  $b$  according to observer  $m$  is defined as the set of coordinate points where  $b$  was observed by  $m$ , i.e.,

$$wl_m(b) := \{p \in \mathbb{Q}^d : W(m, b, p)\}.$$



**AxLinTime** The world-lines of observers are lines and time is elapsing uniformly on them:

$$\forall m, k \in \text{IOb} \quad \text{wl}_m(k) = \{ w_m^k(o) + \lambda \cdot 1_m^k : \lambda \in \mathbb{Q} \} \wedge \\ \forall p, q \in \text{wl}_m(k) \quad \text{time}_k(\text{ev}_m(p), \text{ev}_m(q)) \cdot |1_m^k| = |p - q|.$$

Let us collect the axioms introduced so far in an axiom system:

$$\boxed{\text{Kinem}_0 := \{ \text{AxEOF}, \text{AxSelf}, \text{AxLinTime}, \text{AxEv} \}}$$

Let us note that  $\text{Kinem}_0$  is a general axiom system of kinematics in which no relativistic effect is assumed.  $\text{Kinem}_0$  is a subtheory of Newtonian and relativistic kinematics.

### 3. Geometrical Characterization of TwP

Since the axiom systems we use here deal only with inertial motions of observers, we formulate the inertial version of TwP, which is also called clock paradox in the literature.<sup>||</sup> Logical investigation of the accelerated version of TwP needs a more complex mathematical apparatus, see [14], [28, §4.3], [29, §7]. We also formulate and characterize variants of TwP: one where the stay-at-home twin will be the younger one (Anti-TwP) and another where no differential aging will take place (No-TwP).

To formulate TwP, first we formulate the situations in which it can occur. We say that observer  $m$  observes observers  $a$ ,  $b$  and  $c$  in a **twin paradox situation** at events  $e$ ,  $e_a$  and  $e_c$  iff  $a \in e_a \cap e$ ,  $b \in e_a \cap e_c$ ,  $c \in e \cap e_c$ ,  $b \notin e$  and  $\text{time}_m(e_a) < \text{time}_m(e) < \text{time}_m(e_c)$  or  $\text{time}_m(e_a) > \text{time}_m(e) > \text{time}_m(e_c)$ , see Figure 2. This situation is denoted by  $\text{meetTwP}_m(\widehat{ac}, b)(e_a, e, e_c)$ .

Let  $a, b, c \in \text{IOb}$  and  $e_a, e, e_b \in \text{Ev}$ . Let  $\text{time}(\widehat{ac} < b)(e_a, e, e_b)$  be the abbreviation of  $\text{time}_a(e_a, e) + \text{time}_c(e, e_b) < \text{time}_b(e_a, e_b)$ . The definitions of  $\text{time}(\widehat{ac} = b)(e_a, e, e_b)$  and  $\text{time}(\widehat{ac} > b)(e_a, e, e_b)$  are analogous. Using this notation, we can formulate the twin paradox as follows:

**TwP** Every observer  $m$  observes the twin paradox in every twin paradox situation:

$$\forall m, c, a, b \in \text{IOb} \quad \forall e, e_a, e_c \in \text{Ev}_m \\ \text{meetTwP}_m(\widehat{ac}, b)(e_a, e, e_c) \rightarrow \text{time}(\widehat{ac} < b)(e_a, e, e_c).$$

<sup>||</sup>This inertial version is the one that was formulated by Einstein in his famous 1905 paper, see [8, §4].



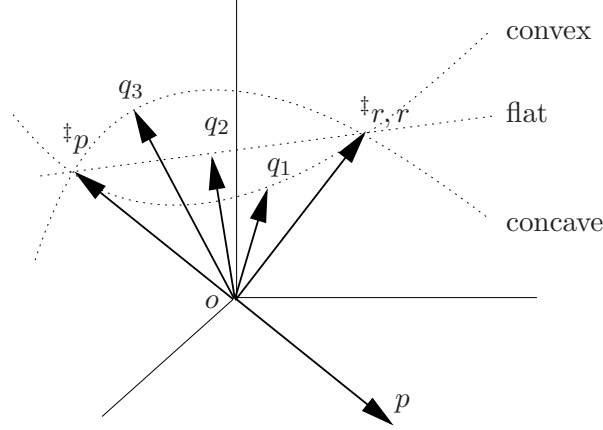


Figure 3. Illustration of relations  $\text{Conv}(p, q_1, r)$ ,  $\text{Bw}(p, q_2, r)$  and  $\text{Conc}(p, q_3, r)$

and  $e_c$  events such that  $\text{meetTwP}_m(\widehat{ac}, b)(e_a, e, e_c)$ . Then

$$\begin{aligned} \text{time}(\widehat{ac} < b)(e_a, e, e_c) &\iff \text{Conv}(\dagger 1_m^a, \dagger 1_m^b, \dagger 1_m^c), \\ \text{time}(\widehat{ac} = b)(e_a, e, e_c) &\iff \text{Bw}(\dagger 1_m^a, \dagger 1_m^b, \dagger 1_m^c), \\ \text{time}(\widehat{ac} > b)(e_a, e, e_c) &\iff \text{Conc}(\dagger 1_m^a, \dagger 1_m^b, \dagger 1_m^c). \end{aligned}$$

PROOF. Let  $m$ ,  $a$ ,  $b$ , and  $c$  be observers and  $e$ ,  $e_a$  and  $e_c$  events such that  $\text{meetTwP}_m(\widehat{ac}, b)(e_a, e, e_c)$ . Let us abbreviate time-unit vectors  $\dagger 1_m^k$  to  $k^\dagger$  throughout this proof. Let  $p = \text{Crd}_m(e_a)$ ,  $q = \text{Crd}_m(e)$  and  $r = \text{Crd}_m(e_c)$ . We have that  $p \neq r$  since  $p_\tau < r_\tau$  or  $r_\tau < p_\tau$ . Therefore, by **AxLinTime**, the triangle  $pqr$  is nondegenerate since  $p, r \in \text{wl}_m(b)$  but  $q \notin \text{wl}_m(b)$ . Let us first show that  $b$  measures the same length of time between  $e_a$  and  $e_c$  as  $a$  and  $c$  together if  $\text{Bw}(a^\dagger, b^\dagger, c^\dagger)$  holds. Let  $s$  be the intersection of line  $pr$  and the line parallel to  $a^\dagger c^\dagger$  through  $q$ , see Figure 2. Since  $\text{Bw}(a^\dagger, b^\dagger, c^\dagger)$  holds, the triangles  $oa^\dagger b^\dagger$  and  $pqs$  are similar; and the triangles  $ob^\dagger c^\dagger$  and  $rsq$  are similar. Thus

$$\frac{|p - q|}{|a^\dagger|} = \frac{|p - s|}{|b^\dagger|} \quad \text{and} \quad \frac{|q - r|}{|c^\dagger|} = \frac{|s - r|}{|b^\dagger|}$$

hold. From which, by **AxLinTime**, it follows that

$$\begin{aligned} \left| \text{time}_a(e_a, e) \right| + \left| \text{time}_c(e, e_c) \right| &= \frac{|p - q|}{|a^\dagger|} + \frac{|q - r|}{|c^\dagger|} \\ &= \frac{|p - s| + |s - r|}{|b^\dagger|} = \frac{|r - p|}{|b^\dagger|} = \left| \text{time}_c(e_a, e_c) \right|. \end{aligned}$$

Hence  $\text{time}(\widehat{ac} = b)(e_a, e, e_c)$  holds if  $\text{Bw}(a^\ddagger, b^\ddagger, c^\ddagger)$ . By **AxLinTime**,  $b$  measures more (less) time between  $e_a$  and  $e_c$  iff his time-unit vector is shorter (longer). Thus we get that  $\text{time}(\widehat{ac} < b)(e_a, e, e_c)$  holds if  $\text{Conv}(a^\ddagger, b^\ddagger, c^\ddagger)$ , and  $\text{time}(\widehat{ac} > b)(e_a, e, e_c)$  holds if  $\text{Conc}(a^\ddagger, b^\ddagger, c^\ddagger)$ . The converse implications also hold since one of the relations **Conv**, **Bw** and **Conc** holds for  $a^\ddagger$ ,  $b^\ddagger$  and  $c^\ddagger$ , and only one of the relations  $\text{time}(\widehat{ac} < b)$ ,  $\text{time}(\widehat{ac} = b)$  and  $\text{time}(\widehat{ac} > b)$  can hold for events  $e_a$ ,  $e$  and  $e_c$ . This completes the proof. ■

A set  $H \subseteq \mathbb{Q}^d$  is called **convex** iff  $\text{Conv}(p, q, r)$  for all  $p, q, r \in H$  for which there is  $\mu \in \mathbb{Q}$  such that  $\text{Bw}(p, \mu q, r)$ . We call  $H$  **flat** or **concave** if  $\text{Conv}(p, q, r)$  is replaced by  $\text{Bw}(q, r, p)$  or  $\text{Conc}(r, p, q)$ , respectively.

**Remark 2.** If there are no  $p, q, r \in H$  for which there is a  $\mu \in \mathbb{Q}^+$  such that  $\text{Bw}(p, \mu q, r)$  holds, then  $H$  is convex, flat and concave at the same time. To avoid these undesired situations, let us call  $H$  **nontrivial** if there are  $p, q, r \in H$  such that  $\text{Bw}(p, \mu q, r)$  holds for a  $\mu \in \mathbb{Q}^+$ . By the respective definitions, it is easy to see that any nontrivial convex (flat, concave) set intersects a halfline at most once.

Let us define the **Minkowski sphere** here as  $MS_m^\ddagger := \{ \ddagger 1_m^k : k \in \text{IOb} \}$ .

**Remark 3.** Convexity as used here is not far from convexity as understood in geometry or in the case of functions. For example, in the models of  $\text{Kinem}_0 + \text{AxThExp}^+$  or  $\text{SpecRel}_d^- + \text{AxThExp}$  (see next sections) the Minkowski Sphere  $MS_m^\ddagger$  is convex in our sense iff the set of points above it  $\{p \in \mathbb{Q}^d : \exists q \in MS_m^\ddagger p_\tau \geq q_\tau\}$  is convex in the geometrical sense.

**Remark 4.** By Remark 2, if  $MS_m^\ddagger$  is a nontrivial convex (flat, concave) set, it intersects a line at most once.

The following is a corollary of Proposition 1.

**Corollary 1.** Assume  $\text{Kinem}_0$ . Then

$$\begin{aligned} \forall m \in \text{IOb } MS_m^\ddagger \text{ is convex} &\implies \text{TwP}, \\ \forall m \in \text{IOb } MS_m^\ddagger \text{ is flat} &\implies \text{noTwP}, \\ \forall m \in \text{IOb } MS_m^\ddagger \text{ is concave} &\implies \text{antiTwP}. \end{aligned}$$

The implications in Corollary 1 cannot be reversed because there may be observers that are not part of any twin paradox situation. We can resolve this problem by using the following axiom to shift observers in order to create twin paradox situations.

**AxShift** If an observer observes another observer with a certain time-unit vector, it also observes still another observer, with the same time-unit vector, at each coordinate point of its coordinate domain:

$$\forall m, k \in \text{IOb} \quad \forall p \in Cd_m \quad \exists h \in \text{IOb} \quad h \in \text{ev}_m(p) \wedge 1_m^k = 1_m^h.$$

Axiom **AxShift** postulates the existence of some observers. Since by observes (bodies) we mean potential observers (potential bodies) these kinds of assumptions are quite natural, see also axioms **AxThExp**<sup>+</sup>, **AxThExp**<sup>\*</sup> and **AxThExp** on pages 14 and 18. Now we can reverse the implications of Corollary 1.

**Theorem 1.** Assume **Kinem**<sub>0</sub> and **AxShift**. Then

$$\begin{aligned} \text{TwP} & \iff \forall m \in \text{IOb} \quad MS_m^\ddagger \text{ is convex,} \\ \text{noTwP} & \iff \forall m \in \text{IOb} \quad MS_m^\ddagger \text{ is flat,} \\ \text{antiTwP} & \iff \forall m \in \text{IOb} \quad MS_m^\ddagger \text{ is concave.} \end{aligned}$$

PROOF. By Corollary 1, we have to prove the “ $\implies$ ” part only. For that, let us take three points  $a'$ ,  $b'$  and  $c'$  from  $MS_m^\ddagger$  for which there is  $\mu \in Q$  satisfying  $\text{Bw}(\ddagger a', \mu b', \ddagger c')$ . If there are no such points,  $MS_m^\ddagger$  is convex, flat and concave at the same time, see Remark 2. Otherwise, by **AxShift** there are observers  $a$ ,  $b$  and  $c$  in a twin paradox situation such that  $1_m^a = a'$ ,  $1_m^b = b'$  and  $1_m^c = c'$ . Thus from Proposition 1 we get that  $MS_m^\ddagger$  has the desired property. ■

In the sections below we will use the following concept. Let  $\Sigma$  and  $\Gamma$  be sets of formulas, and let  $\varphi$  and  $\psi$  be formulas of our language. Then  $\Sigma$  **logically implies**  $\varphi$ , in symbols  $\Sigma \models \varphi$ , iff  $\varphi$  is true in every model of  $\Sigma$ . To simplify our notations, we use the plus sign between formulas and sets of formulas in the following way:  $\Sigma + \Gamma := \Sigma \cup \Gamma$ ,  $\varphi + \psi := \{\varphi, \psi\}$  and  $\Sigma + \varphi := \Sigma \cup \{\varphi\}$ .

**Remark 5.** Let us note that the fewer axioms  $\Sigma$  contains, the stronger the logical implication  $\Sigma \models \varphi$  is, and similarly the more axioms  $\Sigma$  contains the stronger the counterexample  $\Sigma \not\models \varphi$  is.

**Remark 6.** By Gödel’s completeness theorem, all the theorems of this paper remain valid if we replace the relation of logical consequence ( $\models$ ) by the deducibility relation of FOL ( $\vdash$ ).

#### 4. Consequences for Newtonian kinematics

Let us investigate the logical connection between No-TwP and the Newtonian assumption on the absoluteness of time.

**AbsTime** Observers measure the same time elapsing between events:

$$\forall m, k \in \text{IOb} \quad \forall e_1, e_2 \in \text{Ev} \quad \text{time}_m(e_1, e_2) = \text{time}_k(e_1, e_2).$$

To strengthen our axiom system, we introduce two axioms that ensure the existence of several observers.

**AxThExp<sup>+</sup>** Observers can move in any direction at any finite speed:

$$\forall m \in \text{IOb} \quad \forall p, q \in \mathbb{Q}^d \quad p_\tau \neq q_\tau \rightarrow \exists k \in \text{IOb} \quad k \in \text{ev}_m(p) \cap \text{ev}_m(q).$$

This axiom as well as its variants (**AxThExp<sup>\*</sup>** below and **AxThExp** on page 18) are closely related to the assumptions of homogeneity and isotropy of space since in some respect they say that there is no difference between the different points and directions in space. A more experimental version of axiom **AxThExp<sup>+</sup>** is the following:

**AxThExp<sup>\*</sup>** Observers can move in any direction at a speed which is arbitrarily close to any finite speed:

$$\begin{aligned} \forall m \in \text{IOb} \quad \forall p, q \in \mathbb{Q}^d \quad \forall \varepsilon \in \mathbb{Q}^+ \quad p_\tau \neq q_\tau \\ \rightarrow \exists k \in \text{IOb} \quad \exists q' \in \mathbb{Q}^d \quad |q - q'| < \varepsilon \wedge k \in \text{ev}_m(p) \cap \text{ev}_m(q'). \end{aligned}$$

Since the accuracy of an experiment is finite and we can make only finitely many experiments, axiom **AxThExp<sup>\*</sup>** is a more plausible assumption than **AxThExp<sup>+</sup>** from empirical point of view.

By the following theorem, **noTwP** logically implies **AbsTime** if **AxThExp<sup>+</sup>** (and some auxiliary axioms) are assumed; however, if we assume the more experimental axiom **AxThExp<sup>\*</sup>** instead of **AxThExp<sup>+</sup>**, **AbsTime** does not follow from **noTwP**, which is an astonishing fact since it means that without the strong theoretical assumption **AxThExp<sup>+</sup>** we would not be able to conclude that time is absolute in the Newtonian sense even if there were no twin paradox in our world.

**Theorem 2.**

$$\text{AxEOF} + \text{AbsTime} \models \text{noTwP}, \text{ and} \quad (1)$$

$$\text{Kinem}_0 + \text{AxShift} + \text{AxThExp}^+ + \text{noTwP} \models \text{AbsTime}, \text{ but} \quad (2)$$

$$\text{Kinem}_0 + \text{AxShift} + \text{AxThExp}^* + \text{noTwP} \not\models \text{AbsTime}. \quad (3)$$

PROOF. Item (1) is obvious.

To prove (2), let us note that  $MS_m^\ddagger$  is flat by Theorem 1 since **Kinem<sub>0</sub>**, **AxShift** and **noTwP** are assumed. So  $MS_m^\ddagger$  is a subset of a hyperplane. By axiom **AxThExp<sup>+</sup>**,  $MS_m^\ddagger$  intersects any nonhorizontal line. If the hyperplane containing  $MS_m^\ddagger$  were not horizontal, there would be nonhorizontal lines parallel to it. Therefore  $MS_m^\ddagger$  has to be a subset of a horizontal hyperplane. If  $MS_m^\ddagger$  were a proper subset of this hyperplane, there would be nonhorizontal lines not intersecting it. So  $MS_m^\ddagger$  has to be a horizontal hyperplane containing  $\langle 1, 0, \dots, 0 \rangle = 1_m^m$ . Hence the time components of time-unit vectors are the same for every observer. So **AbsTime** follows from the assumptions.

To prove (3), we construct a model in which **Kinem<sub>0</sub>**, **AxShift**, **AxThExp<sup>\*</sup>** and **noTwP** hold, but **AbsTime** does not. Let  $\langle \mathbb{Q}; +, \cdot, \langle \rangle$  be any Euclidean ordered field. Let  $B := \mathbb{Q}^d \times \mathbb{Q}^d$ . Let  $\text{IOb} := \{ \langle p, q \rangle \in B : p_\tau \neq q_\tau \wedge p_\tau - q_\tau \neq p_2 - q_2 \}$ . Let  $MS_{\langle 1, 0 \rangle}^\ddagger := \{ x \in \mathbb{Q}^d : x_\tau - x_2 = 1 \wedge x_\tau > 0 \}$ . Let  $W(\langle 1, 0 \rangle, \langle p, q \rangle, r)$  hold iff  $r$  is in the line through  $p$  and  $q$ . Now the worldview relation is given for observer  $\langle 1, 0 \rangle$ . For any other observer  $\langle p, q \rangle$ , let  $w_{\langle 1, 0 \rangle}^{\langle p, q \rangle}$  be an affine transformation that takes  $o$  to  $p$  while its linear part takes  $1_t$  to  $MS_{\langle 1, 0 \rangle}^\ddagger \cap \{ \lambda(p - q) : \lambda \in \mathbb{Q} \}$ , and leaves the other basis vectors fixed. From these worldview transformations, it is easy to define the worldview relations of other observers, hence our model is given. It is not difficult to see that **Kinem<sub>0</sub>**, **AxShift** and **AxThExp<sup>\*</sup>** are true in this model. Since  $MS_{\langle 1, 0 \rangle}^\ddagger$  is flat and the worldview transformations are affine ones, it is clear that  $MS_m^\ddagger$  is flat for all  $m \in \text{IOb}$ . Hence **noTwP** is also true in this model by Corollary 1. It is easy to see that **AbsTime** implies that  $(1_m^k)_\tau = \pm 1$  for all  $m, k \in \text{IOb}$ . Hence **AbsTime** is not true in this model, as we claimed. ■

## 5. Consequences for special relativity theory

Now we are going to investigate the consequences of Theorem 1 for special relativity. To do so, let us extend our language by a new unary relation **Ph** on **B** for **photons** (light signals) and formulate an axiom on the constancy of the speed of light. For convenience, this speed is chosen to be 1.

**AxPh** For every observer, there is a photon through two coordinate points  $p$  and  $q$  iff the slope of  $p - q$  is 1:

$$\begin{aligned} \forall m \in \text{IOb} \quad \forall p, q \in \mathbb{Q}^d \quad |p_\sigma - q_\sigma| &= |p_\tau - q_\tau| \\ &\leftrightarrow \text{Ph} \cap \text{ev}_m(p) \cap \text{ev}_m(q) \neq \emptyset. \end{aligned}$$

Let us also introduce a symmetry axiom.

**AxSymDist** If events  $e_1$  and  $e_2$  are simultaneous for both the observers  $m$  and  $k$ , then  $m$  and  $k$  agree as to the spatial distance between  $e_1$  and  $e_2$ :

$$\begin{aligned} \forall m, k \in \text{IOb} \quad \forall e_1, e_2 \in \text{Ev} \quad \text{time}_m(e_1, e_2) = \text{time}_k(e_1, e_2) = 0 \\ \rightarrow \text{dist}_m(e_1, e_2) = \text{dist}_k(e_1, e_2), \end{aligned}$$

where the **spatial distance** between events  $e_1$  and  $e_2$  according to observer  $m$ , in symbols  $\text{dist}_m(e_1, e_2)$ , is formulated as  $|\text{Crd}_m(e_1)_\sigma - \text{Crd}_m(e_2)_\sigma|$ .

Let us introduce the following axiom system:

$$\text{SpecRel}_d := \{ \text{AxEOF}, \text{AxSelf}, \text{AxPh}, \text{AxEv}, \text{AxSymDist} \}$$

Now we have a FOL axiom system of special relativity for each natural number  $d \geq 2$ .

To state the Alexandrov-Zeeman theorem generalized for fields, we need a definition. A map  $\tilde{\varphi} : \mathbb{Q}^d \rightarrow \mathbb{Q}^d$  is called a **field-automorphism-induced** iff there is an automorphism  $\varphi$  of the field  $\langle \mathbb{Q}, \cdot, + \rangle$  such that  $\tilde{\varphi}(p) = \langle \varphi(p_1), \dots, \varphi(p_d) \rangle$  for every  $p \in \mathbb{Q}^d$ .

**Theorem 3 (Alexandrov-Zeeman).** Let  $F$  be a field and  $d \geq 3$ . Every bijection from  $F^d$  to  $F^d$  that transforms lines of slope 1 to lines of slope 1 is a Poincaré transformation composed by a dilation and a field-automorphism-induced map.

For the proof of Theorem 3, see [32], [33]. From this theorem we derive that the worldview transformations between observers are Poincaré ones in the models of  $\text{SpecRel}_d$  if  $d \geq 3$ , cf. [3, Theorem 11.11, p.641]. This fact justifies our calling  $\text{SpecRel}_d$  an axiom system of special relativity.

**Theorem 4.** Let  $d \geq 3$ . Let  $m, k \in \text{IOb}$ . Then

1. if **AxEOF**, **AxPh** and **AxEv** are assumed,  $w_m^k$  is a Poincaré transformation composed by a dilation  $D$  and a field-automorphism-induced map  $\tilde{\varphi}$ ;
2. if **AxEOF**, **AxPh**, **AxEv** and **AxSymDist** are assumed,  $w_m^k$  is a Poincaré transformation.

ON THE PROOF. It is not difficult to see that **AxPh** and **AxEv** imply that  $w_m^k$  is a bijection from  $\mathbb{Q}^d$  to  $\mathbb{Q}^d$  that preserves lines of slope 1, see, e.g., [29, Proposition 3.1.3]. Hence Item (1) is a consequence of Theorem 3.

Now let us see why Item (2) is true. By Item (1), it is easy to see that there is a line  $l$  such that both  $l$  and its  $w_m^k$  image are orthogonal to



the time-axis. Thus by **AxSymDist**,  $w_m^k$  restricted to  $l$  is distance preserving. Consequently, both the dilation  $D$  and the field-automorphism-induced map  $\tilde{\varphi}$  in Item (2) have to be the identity map. Hence  $w_m^k$  is a Poincaré transformation. ■

Let us now formulate another famous prediction of relativity.

**SlowTime** Relatively moving observers' clocks slow down:

$$\forall m, k \in \text{IOb} \quad \text{wl}_m(k) \neq \text{wl}_m(m) \rightarrow |(1_m^k)_\tau| > 1.$$

To investigate the logical connection between **SlowTime** and **TwP**, let us also introduce a weakened axiom system of special relativity:

$$\text{SpecRel}_d^- := \{ \text{AxEOF}, \text{AxSelf}, \text{AxPh}, \text{AxEv} \}$$

Let us note that if  $d \geq 3$ ,  $\text{SpecRel}_d^-$  is strong enough to prove the most important predictions of special relativity, such as that moving clocks get out of synchronism, see, e.g., [2]. At the same time,  $\text{SpecRel}_d^-$  is weak enough not to prove every prediction of special relativity. For example, it does not entail **TwP** or **SlowTime**. Thus it is possible to compare these predictions within  $\text{SpecRel}_d^-$ .

To prove a theorem about the logical connection between **SlowTime** and **TwP**, we need the following lemma, which states that the fact that three observers are in a twin paradox situation does not depend on the observer that watches them.

**Lemma 1.** Let  $d \geq 3$ . Assume **AxEOF**, **AxPh**, **AxEv** and **AxLinTime**. Let  $m, a, b, c \in \text{IOb}$  and let  $e_a, e, e_b \in Ev$ . Then

$$\text{meetTwP}_m(\widehat{ac}, b)(e_a, e, e_c) \iff \text{meetTwP}_b(\widehat{ac}, b)(e_a, e, e_c).$$

**PROOF.** By (1) of Theorem 4,  $w_m^b$  is a composition of a Poincaré transformation, a dilation and a field-automorphism-induced map since **AxEOF**, **AxPh** and **AxEv** are assumed. By **AxLinTime**, the field-automorphism is trivial. Hence  $\text{time}_m(e)$  is between  $\text{time}_m(e_a)$  and  $\text{time}_m(e_c)$  iff  $\text{time}_b(e)$  is between  $\text{time}_b(e_a)$  and  $\text{time}_b(e_c)$ . This completes the proof since the other parts of the definition of relation **meetTwP** do not depend on observers  $m$  and  $b$ . ■

We cannot consistently extend our theory  $\text{SpecRel}_d^-$  by axiom **AxThExp**<sup>+</sup> since  $\text{SpecRel}_d^-$  implies the impossibility of faster than light motion of observers if  $d \geq 3$ , see, e.g., [2]. That is why we have to weaken this axiom.

**AxThExp** Observers can move in any direction at any speed slower than 1, i.e., less than the speed of light:

$$\begin{aligned} \forall m \in \text{IOb} \quad \forall p, q \in \mathbb{Q}^d \quad |p_\sigma - q_\sigma| < |p_\tau - q_\tau| \\ \rightarrow \exists k \in \text{IOb} \quad k \in \text{ev}_m(p) \cap \text{ev}_m(q). \end{aligned}$$

The following theorem shows that **SlowTime** is logically stronger than **TwP**.

**Theorem 5.** Let  $d \geq 3$ . Then

$$\text{SpecRel}_d^- + \text{AxLinTime} + \text{SlowTime} \models \text{TwP}, \text{ but} \quad (4)$$

$$\text{SpecRel}_d^- + \text{AxShift} + \text{AxLinTime} + \text{AxThExp} + \text{TwP} \not\models \text{SlowTime}. \quad (5)$$

**PROOF.** Item (4) is clear by Lemma 1.

To prove Item (5), let us construct a model in which **SpecRel**<sub>d</sub><sup>-</sup>, **AxShift**, **AxLinTime**, **AxThExp** and **TwP** hold, but **SlowTime** does not. Let  $\langle \mathbb{Q}; +, \cdot, < \rangle$  be any Euclidean ordered field. Let  $B := \mathbb{Q}^d \times \mathbb{Q}^d$ . Let  $\text{IOb} := \{ \langle p, q \rangle \in B : |p_\sigma - q_\sigma| < |p_\tau - q_\tau| \}$ . It is easy to see that there is a nontrivial convex subset  $M$  of  $\mathbb{Q}^d$  such that  $1_t \in M$  and  $|p_\tau| < 1$  for some  $p \in M$ . Let  $MS_{\langle 1,0 \rangle}^\ddagger$  be such a convex subset of  $\mathbb{Q}^d$ . Let  $W(\langle 1,0 \rangle, \langle p, q \rangle, r)$  hold iff  $r$  is in the line through  $p$  and  $q$ . Now the worldview relation is given for observer  $\langle 1,0 \rangle$ . By Remark 4,  $MS_{\langle 1,0 \rangle}^\ddagger$  intersects a line at most once. For any other observer  $\langle p, q \rangle$ , let  $w_{\langle 1,0 \rangle}^{\langle p, q \rangle}$  be such a composition of a Lorentz transformation, a dilation and a translation which takes  $o$  to  $p$  while its linear part takes  $1_t$  to the unique element of  $MS_{\langle 1,0 \rangle}^\ddagger \cap \{ \lambda(p - q) : \lambda \in \mathbb{Q} \}$ , and leaves the other basis vectors fixed. It is easy to see that there is such a transformation. From these worldview transformations, it is easy to define the worldview relations of the other observers. So the model is given. It is not difficult to see that **SpecRel**<sub>d</sub><sup>-</sup>, **AxShift**, **AxLinTime** and **AxThExp** are true in this model. Since  $MS_{\langle 1,0 \rangle}^\ddagger$  is convex and the worldview transformations are affine ones, it is clear that  $MS_m^\ddagger$  is convex for all  $m \in \text{IOb}$ . Hence **TwP** is also true in this model by Corollary 1. It is clear that **SlowTime** is not true in this model since there is a  $p \in MS_{\langle 1,0 \rangle}^\ddagger$  such that  $|p_\tau| < 1$  (i.e., there is  $k \in \text{IOb}$  such that  $|(1_{\langle 1,0 \rangle}^k)_\tau| < 1$ ); and that completes the proof. ■

Like the similar results of [27] and [28], the following theorem answers Question 4.2.17 of Andr eka–Madar asz–N emeti [1]. It shows that **TwP** is logically weaker than the symmetry axiom of **SpecRel**<sub>d</sub>.

**Theorem 6.** Let  $d \geq 3$ . Then

$$\text{SpecRel}_d^- + \text{AxSymDist} \models \text{TwP}, \text{ but} \quad (6)$$

$$\text{SpecRel}_d^- + \text{AxShift} + \text{AxLinTime} + \text{AxThExp} + \text{TwP} \not\models \text{AxSymDist}. \quad (7)$$

PROOF. By (2) of Theorem 4,  $\text{SpecRel}_d^-$  and  $\text{AxSymDist}$  imply that  $w_m^k$  is a Poincaré transformation for all  $m, k \in \text{IOb}$ . Hence  $MS_m^\dagger \subseteq \{p \in \mathbb{Q}^d : p_\tau^2 - |p_\sigma|^2 = 1 \wedge p_\tau > 0\}$ . Consequently,  $MS_m^\dagger$  is convex. So by Corollary 1,  $\text{TwP}$  follows from  $\text{SpecRel}_d^-$  and  $\text{AxSymDist}$ .

Since  $\text{SpecRel}_d^-$  and  $\text{AxSymDist}$  imply  $\text{SlowTime}$  if  $d \geq 3$ , Item (7) follows from Theorem 5.  $\blacksquare$

It is interesting that  $\text{AxSymDist}$  and  $\text{SlowTime}$  are equivalent in the models of  $\text{SpecRel}_d^-$  (and some auxiliary axioms) if the quantity part is the field of real numbers. However, that the quantity part is the field of real numbers cannot be formulated in any FOL language of spacetime theories. Consequently, nor can Theorem 7, so it cannot be formulated and proved within our FOL frame either.

**Theorem 7.** Let  $d \geq 3$ . Assume  $\text{SpecRel}_d^-$ ,  $\text{AxThExp}$ ,  $\text{AxLinTime}$ ,  $\text{AxShift}$ , and that  $\mathbb{Q}$  is the field of real numbers. Then

$$\text{SlowTime} \iff \text{AxSymDist}. \quad (8)$$

For proof of Theorem 7, see [28, §3]. This theorem is interesting because it shows that assuming only that all moving clocks slow down to some degree implies the exact ratio of the slowing down of moving clocks (since if  $d \geq 3$ ,  $\text{SpecRel}_d^- + \text{AxSymDist}$  implies that the worldview transformations are Poincaré ones, see Theorem 4).

**Question 1.** Does Theorem 7 retain its validity if the assumption that  $\mathbb{Q}$  is the field of real numbers is removed? If not, is it still possible to replace it by a FOL assumption, e.g., by the axiom schema of continuity used in [14], [15], [29, §7.2]?

## 6. Concluding remarks

We have seen that (the inertial version of)  $\text{TwP}$  can be characterized geometrically within a general axiom system of kinematics. We have also seen some surprising consequences of this characterization; in particular, that  $\text{TwP}$  is logically weaker than axiom  $\text{AxSymDist}$  of special relativity as well

as the assumption of the slowing down of moving clocks. A future task is to explore the logical connections between other assumptions and predictions of relativity theories. For example, in [14], [28], [29, §6]  $\text{SpecRel}_d$  is extended to an axiom system  $\text{AccRel}$  logically implying the accelerated version of TwP, but the natural question below, raised by Theorem 6, has not been answered yet.

**Question 2.** Is it possible to weaken  $\text{AxSymDist}$  to TwP in  $\text{AccRel}$  (see, e.g., [29]) without losing the accelerated version of TwP as a consequence? See [14, Question 3.8] and [29, Question 4.5.6].

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