MASTER THESIS

A First Order Logic Investigation of the Twin Paradox and Related Subjects

Gergely Székely

Advisers: Judit X. Madarász
István Németi

Budapest, 2004.

Contents

Chapter 1

Introduction

Rube Walker: "Hey, Yogi, what time is it?"

Yogi Berra: "You mean now?"

In this work, we continue building the theory that Hajnal Andréka, Judit X. Madarász and István Németi began to build with contributions from Attila Andai, Ildikó Sain, Gábor Sági, Csaba Tőke and Sándor Vályi less than a decade ago, cf. [3], [1], [2], [14] or [15].[1] We use first-order logic (FOL) as a framework for studying relativity theory. We have chosen FOL for methodological reasons for the latter, cf., e.g., the appendix "Why FOL?" of [3, pp.1245-1252].

Throughout this work, we concentrate mainly on the twin paradox and related subjects but sometimes we stray away from this topic to peek into some other interesting areas, too. We will discuss several formulations of the twin paradox. They will differ in level of subtlety or faithfulness.

In chapter two, we generalize the frame language that was used before, e.g., in [3] or [1] in the following way: we talk about the dimensions of observers instead of the dimension of space-time. This is the first step toward building a theory where not only the space-time dimension is not fixed but time can be more-than-one dimensional for some observers. We also draw up a conjecture here about this more-than-one time dimensional theory that has connections with the Alexandrov-Zeeman theorem, cf., e.g., [1, p.9], [10, pp.178-182] or Theorem 4.1.1. Moreover, we conjecture that this latter step will lead us to such flexible theories that allow more interesting models for faster-than-light motion than what we have now and also helps us to understand the logical structure of closed time-like curves, i.e. of what is popularly called time travel, cf., e.g., [6, p.261], [7], [8], [12], [16], [20, §14] or [21]. Also in this chapter, we construct some models for our most liked axiom system after introducing some basic definitions, cf. Propositions 2.4.1 and 2.4.3. We also list some important and some less important model construction steps here without intending to be exhaustive.

In chapter three, we formulate our first approximation of the twin paradox in a simple language (like the language used in [3, §1 and 2]) in which we cannot yet talk about accelerated observers. A more sophisticated and more refined formulation of the twin paradox, in the richer language of accelerated observers, comes later in chapter four. Coming back to chapter three, here we examine the connection of the twin paradox with some other axioms about space-time, via stating some theorems that geometrically characterize these axioms and the twin paradox, cf. Theorems 3.1.2, 3.2.2 and 3.2.7, cf. also Figure 3.6. By the use of this theorems, we give more general solutions for some problems of Hajnal Andréka, Judit X. Madarász and István Németi published in [3]

[1] cf. also the cooperation between Gábor Etesi, Mark Hogarth (Cambridge), István Németi and Hajnal Andréka, cf., e.g., [9], [12], [13] and [5].

(cf. questions 4.2.10, 4.2.15, 4.2.16 and 4.2.17 in [3]) than the one we have given in [19]. Also in chapter three, we analyze the logical connections between some natural weakened forms of the twin paradox, cf. Figure 3.6. In Theorem 3.3.1, we show that this form of the twin paradox implies impossibility of the faster-than-light motion for the observers. The conceptual analysis of the twin paradox leads us to take a step toward general relativity by forcing us to extend our investigations to accelerated observers.

In chapter four, first we investigate the possibilities of non-inertial motion in the axiom system $\mathsf{Specrel}_0$ without changing the language. During this investigation, we prove Proposition 4.1.2 in which we characterize all the possible non-inertial motions in the two-dimensional models of $\mathsf{Specrel}_0$. We draw up a conjecture about the existence of two-dimensional models where a stronger version of Einstein's principle of (general) relativity is true in a sense, i.e. all observers see the world the same way, and at the same time there are non-inertially moving observers, too. Then we give general definitions for slower-than-light (STL) and faster-than-light (FTL) set, examine the connection between the STL, time-like and causal curves in Theorem 4.1.4 and we also prove that all the STL motions can be reparametrized to be continuous, cf. Proposition 4.3. Also in this chapter, we expand the language of the theory introduced in chapter one by introducing a new relation symbol distinguishing the accelerated observers from the inertial ones and formulate an axiom, called AxAcc. that gives a connection between the inertial and the accelerated observers. After this, we prove that the twin paradox is true in the models of this accelerated version of our axiom system of special relativity, cf. Theorem 4.3.2. We also draw up conjectures about two model constructions for accelerated observers. Finally, we draw the coordinate system of the uniformly accelerated observer without any further comments.

In the Appendix, we build the analysis tools over arbitrary ordered field that we use in chapter four to prove the theorems about the twin paradox.

Acknowledgments I am grateful to my supervisors Judit X. Madarász and István Németi and to Hajnal Andréka for their helps and inspirations. I also would like to thank András Szilágyi for his invaluable help in the typographical setting of this work and for his LaTeX hints.

Chapter 2

An axiomatization of special relativity in FOL

2.1 Frame language

In this section, we introduce the frame language for the (special) relativity that we are going to use throughout this work. We use the expressions "language", "theory" and "the universe of a model" in the sense of *mathematical logic*, cf., e.g., the logic books quoted in [3]. We will use a similar frame language to that in [1, §1], but we will generalize it a bit. In the frame language, we do not fix the dimension of the space-time but change the Ob and the W relations of [1] for being able to talk about the dimensions of the observers. The dimension of an observer will mean the number of parameters the observer uses for coordinatizing.

Our **frame language** contains the following non-logical symbols:

- unary relation symbols Ob_n for each non negative integer n (for **n-dimensional observers**) and F (for **quantities** which are elements of a Field),
- binary function symbols $+, -, \cdot, /$, constants 0, 1 and a binary relation symbol for $\leq$ (for the operations and the ordering of the ordered field F), and
- an $n+2$-ary relation symbol W_n for each $n \geq 1$ (for the n-dimensional **world-view relations**).

The set of **observers** and **bodies** are defined as $Ob := \cup_{n\geq 1} Ob_n$ and $B := \cup_{n\geq 0} Ob_n$, respectively.[1] The bodies will play the role of the objects of our space-times that will be observed (coordinatized) by the observers by the use of the quantities. Our bodies are basically the same as the "test particles" in some of the literature.

The following three axioms will always be assumed throughout this work and their models are called **frame models**:

AxField A first-order axiom saying that F is a linearly ordered field with the functions $+, -, \cdot, /, 0, 1$ and the relation $\leq$.

AxSqrt Every positive element of the field F has a square root.[2]

AxFrame $W_n \subseteq Ob_n \times B \times F^n$, together with $n \neq k \Rightarrow Ob_n \cap Ob_k = \emptyset$ and $\exists m \;\; m \in Ob$.

[1] We identify relations with the sets defined by them.

[2] We use the symbol F both for the ordered field and for its universe.

In words, $\mathsf{AxFrame}$ is the following three statement: the first argument of W_n is an n-dimensional observer, the second is a body while the other arguments are quantities; the dimension of an observer is unique; and there is at least one observer.

By $\mathsf{AxFrame}$, we can talk about the **dimension of an observer**. The dimension of observer m is defined as: $d(m) := n$ iff $m \in Ob_n$.

CONVENTION 1. We use the validity relation $\models$ in the following way: if $\mathfrak{M}$ is a *frame model* and Σ, Γ are sets of formulas, then $\mathfrak{M} \models \Sigma$ denotes that all formulas in Σ are true in the model $\mathfrak{M}$. In this case, we say that $\mathfrak{M}$ is a **model** of Σ. Similarly, $\Sigma \models \Gamma$ means that all formulas in Γ are true in all *frame models* of Σ. In this case, we say that Γ **follows from** Σ, e.g., $\mathsf{Specrel}_0 \models \mathsf{Specrel}_0^\star$. In the case, when $\Gamma = \{\varphi\}$ for some formula φ, then we simply write the following: $\Sigma \models \varphi$.

CONVENTION 2. We write $W(m, b, p)$ instead of $W_{d(m)}(m, b, p)$; this will not be confusing since $d(m)$ is determined by m.

The hearts of our models are the W_n relations. We use these relations to talk about coordinatizations, by reading $W(m, b, p)$ as "observer m observes body b at coordinate point p". This kind of observation has no connection with seeing via photons, it simply means coordinatizing.

2.2 Basic notation, definitions, and conventions

In this section, we list most of the notation and definitions that we are going to use throughout this work.

We use the symbol $\mathbb{R}$ for the ordered field of the real numbers. When we write $F = \mathbb{R}$, we mean that the ordered field F is isomorphic to the ordered field $\mathbb{R}$.

Let R be a binary relation, i.e. set of ordered pairs. Then the **domain** and **range** of R are denoted by $Dom(R) := \{a : \exists b \ (a, b) \in R\}$ and $Rng(R) := \{b : \exists a \ (a, b) \in R\}$, respectively. We think of a **function** as a special binary relation. The **identity relation** on a set A is $Id_A := \{(a, a) : a \in A\}$. The **inverse** of R is $R^{-1} := \{(b, a) : (a, b) \in R\}$. We call a binary relation R **injective** iff when ordered pairs $(a, c), (b, c)$ are in R, then $a = b$. Notice that R is a function iff R^{-1} is injective. If R and S are two binary relations, then their **composition** $R \circ S$ is defined in the following way: $R \circ S := \{(a, b) : \exists c \ (a, c) \in R \wedge (c, b) \in S\}$.[3] If R is a binary relation and A is an arbitrary set, then the **R-image** of A, in symbols $R[A]$, is defined as the following: $R[A] := \{b : \exists a \in A \ (a, b) \in R\}$, (e.g., $ev_m[tr_m(k)] := \{ev_m(p) : p \in tr_m(k)\}$, cf. ev_m and $tr_m(k)$ below).

The **event** (the set of bodies) seen by observer m at coordinate point $p \in F^{d(m)}$ is:

$$ev_m(p) := \{b \in \mathsf{B} : W(m, b, p)\}.$$

The **coordinate domain** of an observer m is the set of those coordinate points where m sees something:

$$\mathsf{CD}(m) := \{p \in F^{d(m)} : ev_m(p) \neq \emptyset\}.$$

From the definition of the events, a function called the **world-view function** or **event function** arises. This function is $ev_m : \mathsf{CD}(m) \ni p \longmapsto ev_m(p) \in Ev_m$. Notice that the world-view relations and the world-view functions can be reconstructed from each other. The **world-view** of observer m is the set of events seen by him:

$$Ev_m := \{ev_m(p) : p \in \mathsf{CD}(m)\}.$$

[3]Notice that if f, g are functions, then $f \circ g(x) = g(f(x))$.

The **set of all events** is the following:

$$Ev := \bigcup_{m \in Ob} Ev_m.$$

The **life-line (trace)** of body b as seen by observer m is defined as the set of those coordinate points where b was observed by m:

$$tr_m(b) := \{p \in \mathsf{CD}(m) : W(m,b,p)\} = \{p \in \mathsf{CD}(m) : b \in ev_m(p)\}.$$

We say that observers k and k' are **brothers** according to observer m iff their life-lines are the same for m, i.e. $tr_m(k) = tr_m(k')$.

We often talk about observers who observe some bodies somewhere, so the following abbreviations are practical:

$$m \xrightarrow{\odot} b :\Longleftrightarrow \exists p \in \mathsf{CD}(m) \ \ b \in ev_m(p).$$

In this case, we say that observer m **sees** body b. We say that observer m **strongly sees** body b if he sees it more than once, formally:

$$m \xrightarrow{\boxdot} b :\Longleftrightarrow \exists p \neq q \in \mathsf{CD}(m) \ \ b \in ev_m(p) \cap ev_m(q).$$

We abbreviate $m \xrightarrow{\boxdot} k \wedge k \xrightarrow{\boxdot} m$ with $m \xleftrightarrow{\boxdot} k$.

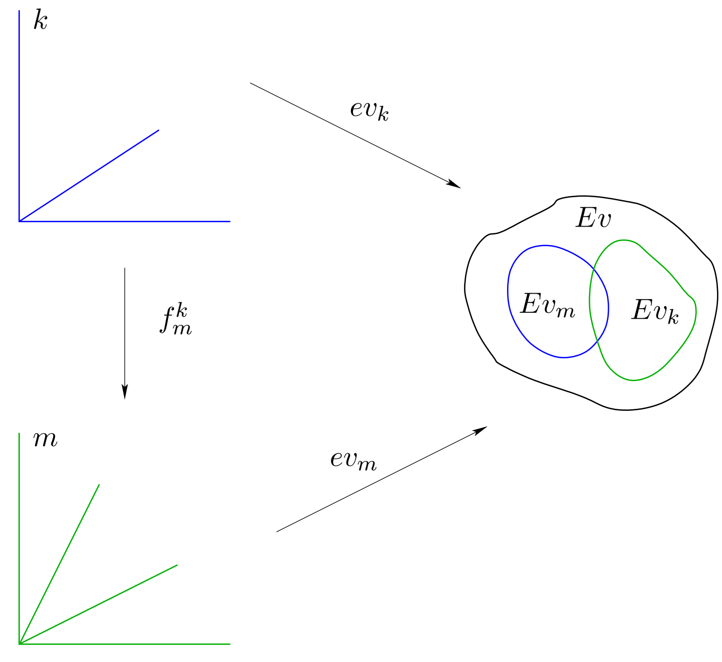

Figure 2.1: for the event function and the world-view transformation.

We are not only interested in coordinate-domains but also in the relations between them. Thus the next definition is fundamental. The **world-view transformation** between two observers m and k is defined as:

$$f^k_m := \{(p,q) \in \mathsf{CD}(k) \times \mathsf{CD}(m) : ev_k(p) = ev_m(q)\},$$

i.e. two coordinate points respectively from the observers k and m's coordinate domains are in this relation if both k and m see the same event at these points, cf. Figure 2.1. Notice that

$f_m^k = ev_k \circ ev_m^{-1}$ as suggested by the commutative triangle in Figure 2.1. Also notice that the world-view transformation is only a binary relation in this general situation but some weak assumptions can turn it into an injective function (e.g., AxPh or $\mathsf{AxPh}^\star$ or even $\mathsf{AxPh_0}$ is enough, see below).

Convention 3. Whenever we write "$f_m^k(p)$", we mean that there is a unique $q \in \mathsf{CD}(m)$ such that $(p,q) \in f_m^k$, and $f_m^k(p)$ denotes this unique q.[4] This convention is very convenient when we are working in one of those axiom systems where the world-view transformation is not a function, because it "turns the world-view transformation into a function in the named point".

Notice that $Id_{\mathsf{CD}(m)} \subseteq f_m^m$ and $f_m^k = (f_k^m)^{-1}$ always hold, while $Id_{\mathsf{CD}(m)} = f_m^m$ holds iff for all observers m the f_m^m is a function or iff for all observers m and k the f_m^k's are functions. The inclusion $f_m^k \circ f_k^m \supseteq Id_{Dom(f_m^k)}$ is also always true since $f_m^k = (f_k^m)^{-1}$, while $f_m^k \circ f_k^m \subseteq Id_{Dom(f_m^k)}$ is equivalent with the statement that f_k^m is a function. Furthermore, $f_h^k \circ f_m^h \subseteq f_m^k$ is always true, while the converse inclusion is equivalent with the statement that h sees all the events that are seen both by k and m (i.e. $ev_h \subseteq ev_k \cap ev_m$, cf. AxEv below). The inclusion $f_m^k[tr_k(b)] \subseteq tr_m(b)$ also follows by the definitions.

Let $p \in F^n$ and let us assume that $p = (p_1, \ldots, p_n)$. We use the notation $p_t := p_1$ for the **time component** of p and $p_s := (p_2, \ldots, p_n)$ for the **space component** of p. The **Euclidean-length** of p is defined as $|p| := \sqrt{p_1^2 + \ldots + p_n^2}$ and the **time-unit vector** as $1_t := (1, 0, \ldots, 0)$. We use the symbol o for the **origin** $(0, \ldots, 0)$.
The **straight-line** through $p, q \in F^n$ is:

$$pq := \{q + \lambda(p - q) : \lambda \in F\}.$$

Notice that $pp = \{p\}$. The set of straight-lines is denoted as:

$$\mathsf{Lines}_n := \{pq : p \neq q \wedge p, q \in F^n\}.$$

If it is not confusing, we omit the subscript of Lines_n.

Let $p, q \in F^n$. Then

$$\mathsf{slope}(p) := \begin{cases} \frac{|p_s|}{|p_t|} & \text{if } p_t \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

and $\mathsf{slope}(pq) := \mathsf{slope}(p - q)$.

Convention 4. In the case $p = (p_1, \ldots, p_k) \in F^k$ and $q = (q_1, \ldots, q_n) \in F^n$, we use the symbolism (p, q) for $(p_1, \ldots, p_k, q_1, \ldots, q_n) \in F^{n+k}$ throughout this work.

The n-dimensional **time-axis** is defined as:

$$\bar{t}_n := \{(t, o) \in F^n : t \in F\}.$$

If it is not confusing, we omit the subscript of $\bar{t}_n$.

Let $F^+ := \{x \in F : x > 0\}$ denote the positive members of F. The (open) **ball** with center $p \in F^n$ and radius $\varepsilon \in F^+$ is the following:

$$B_\varepsilon(p) := \{q \in F^n : |p - q| < \varepsilon\}.$$

The (open) **punctured ball** with center $p \in F^n$ and radius $\varepsilon \in F^+$ is the following:

$$B_\varepsilon^\circ(p) := \{q \in F^n : 0 < |p - q| < \varepsilon\}.$$

[4] Sometimes this convention appears indirectly in some relations defined from the world-view relation, (cf., e.g., $\mathsf{AxLinTime}$, Time_m^k, 1_m^k).

2.3 Some axioms and axiom systems

In this section, we introduce some of the axioms and axiom systems that we work with. The following natural axiom goes back to Galileo Galilei and even to the Norman-French Oresme of around 1350, cf., e.g., [1, p.23, §5]. It simply states that each observer thinks that he rests in the origin of the space part of his coordinate system.

AxSelf The trace of any observer in his own coordinate domain is the time-axis:

$$\forall m \in Ob \;\; tr_m(m) = \bar{t}_{d(m)}.$$

Later, when we are generalizing toward accelerated observers, we will use the following localized version of AxSelf:

$\mathsf{AxSelf}^{\mathsf{loc}}$ The trace of any observer is the intersection of his coordinate domain and the time-axis:

$$\forall m \in Ob \;\; tr_m(m) = \bar{t}_{d(m)} \cap \mathsf{CD}(m).$$

The next axiom is not as natural as the first one but useful to assume it sometimes since it is required for the group property of the world-view transformations. It is also assumed in the other approaches, e.g., in Minkowski geometry, cf., e.g. [14].

AxEv All observers see the same events:

$$\forall m, k \in Ob \;\; Ev_m = Ev_k.$$

Notice that from AxEv it follows that $Dom(f^k_m) = \mathsf{CD}(k)$ and $Rng(f^k_m) = \mathsf{CD}(m)$.

A bit more than a hundred years ago, all the problems started when Michelson and Morley got as the result of their experiment that there are some objects called photons whose speeds are independent of who measures it.[5] For being able to talk about these objects, we introduce a unary relation Ph on the set B of bodies for **Photons**. In the following axiom, we postulate that these bodies are acting like the "real" photons according to the Michelson-Morley experiment and that from every point it is possible to send a photon in every direction. For convenience, we choose 1 for the speed of photons.

AxPh The traces of the photons are exactly the straight-lines with slope 1:

$$\forall m \in Ob \;\; \{tr_m(ph) : ph \in Ph\} = \{l \in \mathsf{Lines}_{d(m)} : \mathsf{slope}(l) = 1\}.$$

We note that, from AxPh, it follows that $\mathsf{CD}(m) = F^{d(m)}$ and the world-view transformation f^k_m is an injective function for every m and k, see the proof of Proposition 2.3.1 below for details.

Later, while axiomatizing accelerated observers, we will need variants of our present axioms which do *not* imply $\mathsf{CD}(m) = F^{d(m)}$ for accelerated m. In later generalizations, we will want to replace AxPh, as in the case of AxSelf, with a localized version, e.g., $\mathsf{AxPh}^{\mathsf{loc}}_0$ below, cf. also [15].

We call the axiom system that contains these three axioms $\mathsf{Specrel}_0$ because the most characteristic predictions of special relativity theory can be derived from it, i.e. "relatively moving clocks slow down", "relatively moving space ships shrink" and "relatively moving pairs of clocks get out of synchronism", cf. [1, §1].

$$\mathsf{Specrel}_0 := \{\mathsf{AxSelf}, \mathsf{AxPh}, \mathsf{AxEv}\}.$$

We call a set **photon-line** if it is the trace of a photon. Notice that photon-lines are not supposed to be straight-lines in general but if we assume AxPh, then they must be straight-lines with slope 1.

The following proposition says that AxPh and AxEv already imply that there are no different dimensional observers.

[5] Actually, this result could have been anticipated on the basis of earlier results on electromagnetism but we ignore this aspect of history for simplicity here.

Proposition 2.3.1. $\mathsf{AxPh}, \mathsf{AxEv} \models \forall m, k \in Ob\ \ d(m) = d(k)$.

proof. The statement is clear if there is an observer, say m, whose dimension is one or two, i.e. $d(m) = 1$ or $d(m) = 2$ because of the followings: A one-dimensional observer cannot see any photon because there is no straight-line with slope 1 in his coordinate system. A two-dimensional observer cannot see more than two different photon-lines through the same point but he sees photons. A more-than-two dimensional observer sees more than two different photon-lines through the same point. From AxEv it follows that if one observer sees two photons on different life-lines, then all observers see them on different life-lines.

In the case when $d(m), d(k) > 2$, the world-view transformation f^k_m is a bijection from $F^{d(m)}$ to $F^{d(k)}$ which preserves the photon-lines (i.e. the f^k_m image of a photon-line is a photon-line) because of the following: f^k_m is injective because for any two distinct points p and q in $F^{d(m)}$ there is a photon-line which contains exactly one of them. Furthermore, $Dom(f^k_m) = F^{d(k)}$ and $Rng(f^k_m) = F^{d(m)}$ as we mentioned before.

The set of those points that cannot be reached from a fixed photon-line by photon-lines forms a hyperplane.[6] From the definition of these hyperplanes, it is clear that they have to be preserved by the world-view transformations. In different dimensions, different many of these hyperplanes are needed to get a point by intersection. Thus $d(m)$ and $d(k)$ must be the same. ■

Until now, we have assumed that an observer m uses exactly one dimension for coordinatizing time. We could introduce the following flexibility here: An observer m uses, say, $t(m)$ dimensions for coordinatizing time where $0 \le t(m) \le d(m)$. In this case for a point $p \in \mathsf{CD}(m)$, we define $p_t := \big(p_1, \ldots, p_{t(m)}\big)$ and $p_s := \big(p_{t(m)+1}, \ldots, p_{d(m)}\big)$. This will influence the meaning of slope and thus the meaning of the photon axiom, too.

QUESTION 1. Does Proposition 2.3.1 remain true if we allow more than one dimensions for the observers for coordinatizing time?

We say that a **frame model is n-dimensional** if every observer in it is n-dimensional. So from Proposition 2.3.1, it follows that every model of $\mathsf{Specrel}_0$ is n-dimensional for some natural number n. Therefore it is clear that we have to weaken $\mathsf{Specrel}_0$ if we want different dimensional observers. Now we do so:[7]

The following axiom is a very natural weakened form of AxEv:

AxEvTr An observer observes all the events he was observed in by some other observer:

$$\forall m, k \in Ob\ \ \forall p \in tr_m(k)\ \exists q \in \mathsf{CD}(k)\ \ ev_m(p) = ev_k(q).$$

Notice that AxEvTr is equivalent with the following: $f^k_m[tr_k(k)] = tr_m(k)$ while the converse inclusion is true in all frame models.

There is a similar weakened form for AxPh:

AxPhTr The traces of the photons which intersect the time-axis are exactly the straight-lines which intersect the time-axis with slope 1:

$$\forall m \in Ob\ \ \{tr_m(ph) : ph \in Ph \land tr_m(m) \cap tr_m(ph) \neq \emptyset\} = \{l \in \mathsf{Lines}_{d(m)} : \mathsf{slope}(l) = 1 \land tr_m(m) \cap l \neq \emptyset\}.$$

The motivation for the following axiom is that in our intuitive image of the models where different dimensional observers are allowed a photon can be seen only for a moment for some observers.

[6] This hyperplane is the Minkowski-orthogonal one to the fixed photon-line.

[7] We list more axioms here than we use in this present work for illustrating what kind of natural possible weakened forms of the axioms may arise.

$\mathsf{AxPh}^\star$ If some observer sees a photon at least twice, then he sees it on a straight-line with slope 1 and every observer sees a photon on each straight-line with slope 1:

$$\forall m \in Ob\ \ \{tr_m(ph) : ph \in Ph \wedge m \xrightarrow{\boxdot} ph\} = \{l \in \mathsf{Lines}_{d(m)} : \mathsf{slope}(l) = 1\}.$$

There is a weakened form of this axiom which requires only that the traces of the photons are subsets of straight-lines with slope 1 and for every two different points if there is a straight-line with slope 1 through them, then there is a photon-line through them.

AxPh_0 Through two different points there is a photon-line iff the slope of the straight-line through these two points is 1:

$$\forall m \in Ob\ \ \forall p \neq q \in F^{d(m)}\ \ \big[\exists ph \in Ph\ \ ph \in ev_m(p) \cap ev_m(q)\big] \iff \mathsf{slope}(pq) = 1.$$

Notice that from AxPh_0 it follows that $\mathsf{CD}(m) = F^{d(m)}$. The following weakened form of AxPh_0 is a step toward general relativity where we do not want the observers to use the whole $F^{d(m)}$ for coordinate domain.

$\mathsf{AxPh}_0^{\mathsf{loc}}$ Through two different points from the coordinate domain of an observer there is a photon-line iff the slope of the straight-line through these two points is 1:

$$\forall m \in Ob\ \ \forall p \neq q \in \mathsf{CD}(m)\ \ \big[\exists ph \in Ph\ \ ph \in ev_m(p) \cap ev_m(q)\big] \iff \mathsf{slope}(pq) = 1.$$

The weakest and therefore the less powerful of our photon axioms is the following one:

AxPh_{00} The slope of a straight-line through two points where the same photon was observed by observer m is 1:

$$\forall m \in Ob\ \ \forall p \neq q \in \mathsf{CD}(m)\ \ \big[\exists ph \in Ph\ \ ph \in ev_m(p) \cap ev_m(q)\big] \Longrightarrow \mathsf{slope}(pq) = 1.$$

Notice that AxPh_{00} is the only photon axiom that does not require the existence of photons.

Our favorite collection of the weakened axioms is:

$$\mathsf{Specrel}_0^\star := \{\mathsf{AxSelf}, \mathsf{AxPh}^\star, \mathsf{AxEvTr}\}.$$

We note that in the models of $\mathsf{Specrel}_0^\star$ the dimensions of the observers are not necessarily the same. This follows from Proposition 2.4.1.

The **time-unit** of observer k seen by observer m is defined as:

$$1_m^k := f_m^k(1_t) - f_m^k(o).$$

The following axiom roughly says that if an observer m strongly sees another observer k, then the restriction of the world-view transformation f_m^k to the time-axis is linear.

$\mathsf{AxLinTime}$ If observer m strongly sees observer k, then he thinks that the time is passing uniformly for k:

$$\forall m, k \in Ob\ \ \forall \lambda \in F\ \ m \xrightarrow{\boxdot} k \Longrightarrow f_m^k(\lambda 1_t) - f_m^k(o) = \lambda 1_m^k.$$

Remark 1. Notice that only by assuming $\mathsf{AxLinTime}$ the following statements are hold: if observer m strongly sees observer k, then $\bar{t} \subseteq Dom(f_m^k)$, the restriction of f_m^k to $\bar{t}$ is a function and 1_m^k exists by Convention 3.

Finally, let us introduce a very weak axiom system of kinematics here:

$$\mathsf{Kinem}_0 := \{\mathsf{AxSelf}, \mathsf{AxEvTr}, \mathsf{AxLinTime}\}.$$

2.4 Important models and model construction steps and their relations with the axioms

In this section, we list some of the most important models and a pack of construction steps that help us in constructing new frame models from other fame models or transforming frame models into each other. We also examine the question "under which conditions do these construction steps preserve our axioms".

A **model construction step** is a partial operation that creates a new frame model form one or more given frame models. We say that a model construction step **preserves** an axiom when the following holds: if the axiom was valid in the models before applying the construction step, then it remains valid in the newly created model, too.

There is a so-called **base-model**, over any ordered field F, for $\mathsf{Specrel}_0$ and $\mathsf{Specrel}_0^\star$ where there is only one observer, called **base-observer**, whose life-line is the time-axis and who sees a photon on every one-sloped straight-line.

In the definition of the construction step called the **disjoint union of frame models**, we take the disjoint union of the bodies in some frame models over the *same* field. The construction of the new world-view relation is straightforward for this operation. Notice that the models of $\mathsf{Specrel}_0^\star$ are closed under the disjoint union operation, i.e. the disjoint union construction step preserves the axiom system $\mathsf{Specrel}_0^\star$.

We call a frame model **observationally connected** iff for every two events $\alpha, \beta \in Ev$ there is a sequence $m_1 \dots m_n$ of observers such that $\alpha \in Ev_{m_1}$, $\beta \in Ev_{m_n}$ and $Ev_{m_k} \cap Ev_{m_{k+1}} \neq \emptyset$ for each $1 \leq k < n$.

It is quite natural to try to add (or remove) observers to (from) the models while trying to preserve the axioms (or most of them). The way of changing the world-view relations for **removing bodies** or even **observers** is obvious. If the removed body is not a photon or the only observer in the model, then this operation does not ruin the validity of our axioms. The construction step for **adding bodies** or even **observers** is also free from any problems if we do not want to preserve any of our axioms. This kind of construction begins to be difficult when we want some of our axioms to be preserved. It has been proved that we can add an observer, say k, to any straight-line with slope less than 1 in another observer's, say m's, coordinate domain such that we do not lose the validity of $\mathsf{Specrel}_0$. In this construction, we can freely choose two points from the named straight-line both for $f_m^k(o)$ and $f_m^k(1_t)$, cf. [3, §3.5]. There are some other natural observer-adding construction steps, too, cf. the radar construction or the tangent construction on page 35.

If H is a subset of $\mathsf{CD}(m)$, we can change m's coordinate domain to H by restricting the event function ev_m to H, cf. localizing in [4, §4]. This construction step is called **restriction of the coordinate domain**. It does not necessarily preserve AxPh or $\mathsf{AxPh}^\star$. It preserves AxSelf iff the time-axis is a subset of H, and in this situation the restriction does not screw up AxEvTr either.

Proposition 2.4.1. There is an observationally connected model of $\mathsf{Specrel}_0^\star$ where there is no maximal dimensional observer.

proof. If H is a linear subspace of $F^{d(m)}$ which contains the time-axis and we change the dimension of m to the dimension of H, then the restriction of m's coordinate domain to H preserves $\mathsf{AxPh}^\star$ as well as AxSelf and AxEvTr. By this kinds of restrictions, we can construct models for $\mathsf{Specrel}_0^\star$ which are not models of $\mathsf{Specrel}_0$. Then we can take an infinite chain[8] of these kinds of models of $\mathsf{Specrel}_0^\star$ over the same ordered field such that the union of this chain is a model of $\mathsf{Specrel}_0^\star$ which is observationally connected and there is no maximal dimensional observer in it. ∎

[8] Let $\mathfrak{M}_i$ be a sequence of frame models. We call $\mathfrak{M}_i$ a **chain** if for all $i < j$, $\mathfrak{M}_i$ is a submodel of $\mathfrak{M}_j$.

The construction step called **gluing** is the following one: we take the disjoint copies of two frame models over the same ordered field and a subset of the events in each with a fixed bijection between these two sets; if the event $ev_m(p)$ is in one of the two sets for some observer m and coordinate point p, then we change the image of m's world-view function in p to the union of this event and its bijective image.

We say that a body is **faster-than-light** (FTL) if its trace is a greater-than-one sloped straight-line. We abbreviate the statement "there is no faster-than-light observer" by noFTL. Later in this work, we will generalize the definition of being FTL for arbitrary sets.

The following theorem states that the FTL motion is not allowed in the more-than-two dimensional models of $\mathsf{Specrel}_0$ while in the two-dimensional models it is, cf. [3, Theorem 3.4.1.]:

Theorem 2.4.2 (Madarász).

(i) $\mathsf{Specrel}_0 \cup \{\forall m \in Ob\ \ d(m) > 2\} \models \mathsf{noFTL}$

(ii) $\mathsf{Specrel}_0 \cup \{\forall m \in Ob\ \ d(m) = 2\} \not\models \mathsf{noFTL}$.

The following proposition shows that there are more possibilities for FTL motion in the models of $\mathsf{Specrel}_0^\star$ than in the models of $\mathsf{Specrel}_0$:

Proposition 2.4.3. FTL motion is allowed in the models of $\mathsf{Specrel}_0^\star$ not only for two-dimensional observers.

proof. Let us take two more-than-two dimensional base-models over the same field and a plane which contains the time-axis in each base-observer's coordinate domain. Let us also take a linear (or an affine) isomorphism between these planes which takes the time-axis to a FTL straight-line and takes one-sloped straight-lines into one-sloped straight-lines. These planes and the isomorphism between them determine two subsets of the events in each base-model and a bijection between them. If we glue these two models through this bijection we get a model of $\mathsf{Specrel}_0^\star$ where there are two observers who move FTL according to each other. ■

CONJECTURE 1. If an observer k is FTL according to m in a model of $\mathsf{Specrel}_0^\star$, then $Dom(f_k^m)$ cannot contain a three dimensional affine subspace which contains both the time-axis and the life-line of k. We also conjecture that this can be proved by creating a three dimensional model for $\mathsf{Specrel}_0$ from any counterexample of the conjecture where there is a faster-than-light observer but this is impossible by Theorem 2.4.2.

Linking bodies together is the operation of changing a set $H \subseteq B$ of bodies to one body $b \in B$ in the following way: if $W(m, p, b')$ holds for some $b' \in H$, then we change this b' to b in W. This is useful when we want the photons whose traces intersect in more than one points be the same, e.g., after gluing. Moreover, if we take a frame model where AxPh_0 is true and we link all the photons that intersect the same one-sloped straight-line more-than-once, then we get a model where $\mathsf{AxPh}^\star$ is true. Thus the theorems whose statements are insensible for this latter operation (e.g., there are no photon-lines in the statement) and provable by using $\mathsf{AxPh}^\star$ are provable from AxPh_0, too, cf., e.g., Proposition 2.3.1.

The following axiom is related to Occam's Razor:

AxOccam Every body is observable by some observer, formally:

$$\forall b \in B\ \exists m \in Ob\ \ m \xrightarrow{\odot} b.$$

We call **Occamization** the construction step that removes all the bodies from a frame model that are not observed by some observer. This construction step might be useful when we would like to clear the mess that some other constitution steps might have left behind.

Chapter 3

Twin paradox without accelerated observers

3.1 Characterization of the twin paradox

In this section, we formulate and give a characterization for the inertial version of the twin paradox.

The *twin paradox* is an often quoted paradigmatic effect of relativity theory that is based on the following thought experiment: Let us imagine two twin siblings, called Amelia and Immanuel, and let us imagine that Amelia is an astronaut who travels (and accelerates) a lot through space while Immanuel is a scientist who works at home and does not travel at all. The twin paradox is the surprising statement that says when Amelia returns to the Earth and meets Immanuel she is *younger* than her *twin brother*. To formulate this phenomenon, first, we formulate that two bodies, say a and b, **meet** at coordinate point p according to observer m:

$$\mathsf{meet}^p_m(a,b) :\iff W(m,a,p) \wedge W(m,b,p) \iff \{a,b\} \subseteq ev_m(p).$$

The second thing that has to be formulated is the elapsed time measured by observers between events according to another observer. Since we axiomatize relativity theory in an observationally oriented way we define the time difference between coordinate points instead of events. If m and k are observers, then the **time measured** by k between $p, q \in \mathsf{CD}(m)$ is defined as:

$$\mathsf{Time}^m_k(p,q) := f^m_k(q)_t - f^m_k(p)_t.$$

Notice that $\mathsf{Time}^m_k(p,q)$ is independent from the choice of observer m in the following sense: assuming $p, q \in Dom(f^m_k) \cap Dom(f^m_h)$ and f^m_k and f^m_h are functions both in p and q,

$$\mathsf{Time}^m_k(p,q) = \mathsf{Time}^h_k\big(f^m_h(p), f^m_h(q)\big).$$

Now we are able to formulate the twin paradox in our language. We formulate it in a bit more general situation than mentioned above because we also would like to talk about that some observer sees this phenomenon. So two observers a and b are in **twin paradox relation** at coordinate points p and q in observer m's coordinate domain if b measures more time between p and q than a:

$$\mathsf{Twp}_m(a<b)(p,q) :\iff \left|\mathsf{Time}^m_a(p,q)\right| < \left|\mathsf{Time}^m_b(p,q)\right|.$$

We will give a geometrical characterization to the twin paradox in the axiom system Kinem_0 but before doing so we show some easy but useful consequences of Kinem_0.

The following lemma states that, in the models of Kinem_0, if an observer m strongly sees observer k then the life-line of k is a straight-line in the coordinate domain of m and the word-view transformation is a bijection between the life-lines of k according to m and k, respectively.[1]

Proposition 3.1.1. $\mathsf{Kinem}_0 \models \forall m, k \in Ob \quad \big[m \xrightarrow{\boxdot} k \Longrightarrow tr_m(k) \in \mathsf{Lines}_{d(m)}$ and f_m^k is a bijection between $tr_k(k)$ and $tr_m(k)\big]$.

proof. Let m and k be observers such that m strongly sees k. By $\mathsf{AxLinTime}$, $f_m^k(\lambda 1_t) - f_m^k(o) = \lambda 1_m^k$. Hence $\bar{t} \subseteq Dom(f_m^k)$, $f_m^k[\bar{t}]$ is a straight-line (i.e. the line $\lambda 1_m^k + f_m^k(o)$) and f_m^k is a bijection between $\bar{t}$ and $f_m^k[\bar{t}]$. By AxSelf and AxEvTr, respectively, $tr_k(k) = \bar{t}$ and $tr_m(k) = f_m^k[tr_k(k)]$, cf. remarks below AxEvTr on page 9. Thus $tr_m(k) = f_m^k[tr_k(k)] = f_m^k[\bar{t}\,]$ is a straight-line and f_m^k is a bijection between $tr_k(k) = \bar{t}$ and $tr_m(k) = f_m^k[\bar{t}]$. ■

Since the life-lines of the observers are straight-lines, if two observers meet in two different coordinate points then their life-lines have to be the same. Therefore we have to change the twin paradox relation if we want to treat the twin paradox in Kinem_0. We do this by replacing the accelerating twin with two inertial ones, i.e. we replace the traveling twin with a leaving and an approaching one.[2] So we say that observer m sees observers a, b, c in **twin paradox situation** in coordinate points p, q, r if observers a, b meet at q, observers b, c meet at r, observers a, c meet at p and p, q, r are not collinear and the time coordinates of p, q, r are in this order, i.e. $p_t < q_t < r_t$, see Figure 3.1;

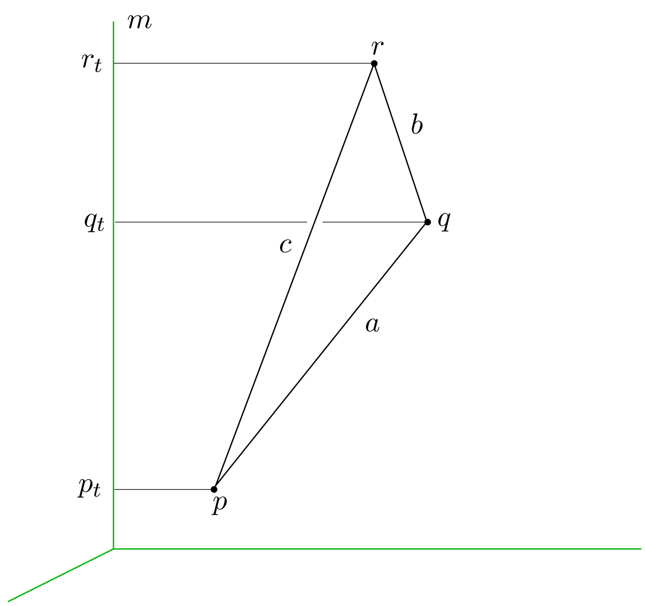

Figure 3.1: for $\mathsf{meetTwp}_m(\widehat{ab}, c)(p, q, r)$.

$$\mathsf{meetTwp}_m(\widehat{ab}, c)(p, q, r) :\Longleftrightarrow \mathsf{meet}_m^q(a, b) \wedge \mathsf{meet}_m^r(b, c) \wedge \mathsf{meet}_m^p(a, c) \wedge p_t < q_t < r_t \wedge q \notin pr.$$

We say that the (**inertial-**)**twin paradox** is true for observers a, b, c at coordinate points p, q, r according to m if c measures more time between p and r than the sum of what a measures

[1]We say that a binary relation R is a bijection between two set A and B iff $A \subseteq Dom(R)$, $B \subseteq Rng(R)$ and $R \cap (A \times B)$ is a bijective function.

[2]This is the common trick of the literature for talking about the twin paradox in special relativity.

between p and q and what b measures between q and r:

$$\mathsf{Twp}_m(\widehat{ab} < c)(p,q,r) := \left|\mathsf{Time}_a^m(p,q)\right| + \left|\mathsf{Time}_b^m(q,r)\right| < \left|\mathsf{Time}_c^m(p,r)\right|.$$

AxTwp Every observer observes the twin paradox in all twin paradox situations:

$$\forall m,a,b,c \in Ob \;\; \forall p,q,r \in \mathsf{CD}(m) \;\; \mathsf{meetTwp}_m(\widehat{ab},c)(p,q,r) \Longrightarrow \mathsf{Twp}_m(\widehat{ab} < c)(p,q,r).$$

We say that observers m and k are **well-configured** in coordinate point p iff they see the same event in p, i.e. $ev_m(p) = ev_k(p)$. If we simply say that m and k are well-configured, we mean that they are well-configured in the origin. Notice that well-configuredness is an equivalence relation on the set of observers.

The (**time-**)**Minkowski-sphere** of an observer m is the set of the time-units of observers well-configured to m:

$$MS_m^t := \{1_m^k : k \in Ob \wedge 1_m^k \text{ exists} \wedge ev_m(o) = ev_k(o)\}.$$

AxDisplace If m is an observer who strongly sees observer k and p is an arbitrary point of m's coordinate domain, then there is another observer h whose time-unit is the same as k's and who sees the same event in o that m sees in p:

$$\forall m,k \in Ob \;\; \forall p \in \mathsf{CD}(m) \;\; m \xrightarrow{\boxdot} k \Longrightarrow \exists h \in Ob \;\; 1_m^k = 1_m^h \wedge ev_m(p) = ev_h(o).$$

Notice that if observer m strongly sees observer k, then the statement $1_m^k \in MS_m^t$ follows from AxDisplace since there is an observer who is well-configured to m and whose time-unit is the same as k's (according to m).

For convenience, we introduce the following notation:

$$p^+ := \begin{cases} (p_t, \ldots, p_n) & \text{if } \; p_t \geq 0 \\ (-p_t, \ldots, -p_n) & \text{if } \; p_t < 0. \end{cases}$$

Assume that $p,q,r \in F^n$. We write $\mathsf{convex(p,q,r)}$ iff o,p,q,r are coplanar and q^+ is in the interior of the triangle op^+r^+, formally: $q^+ \in \{\lambda p^+ + \mu r^+ : 0 < \lambda, \mu \wedge \lambda + \mu < 1\}$ and $or \neq op$. A set $H \subseteq F^n$ is called ***convex*** if $\forall p,q,r \in H$ if op, oq, or are distinct coplanar straight-lines, then $\mathsf{convex}(p,q,r)$ or $\mathsf{convex}(q,r,p)$ or $\mathsf{convex}(r,p,q)$.

The following theorem gives the promised geometrical characterization for the twin paradox axiom and answers the question 4.2.10 in [3].

Theorem 3.1.2.

$$\mathsf{Kinem_0} \cup \{\mathsf{AxDisplace}\} \models \mathsf{AxTwp} \Longleftrightarrow \forall m \in Ob \;\; MS_m^t \; \text{ is } \textit{convex}. \text{ Therefore}$$

$$\mathsf{Specrel_0^\star} \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}\} \models \mathsf{AxTwp} \Longleftrightarrow \forall m \in Ob \;\; MS_m^t \; \text{ is } \textit{convex}.$$

proof. The heart of this proof is the following lemma:

Lemma 3.1.3.

$$\mathsf{Kinem_0} \models \mathsf{meetTwp}_m(\widehat{ab},c)(p,q,r) \Longrightarrow \left[\mathsf{Twp}_m(\widehat{ab} < c)(p,q,r) \Longleftrightarrow \mathsf{convex}(1_m^a, 1_m^c, 1_m^b)\right].$$

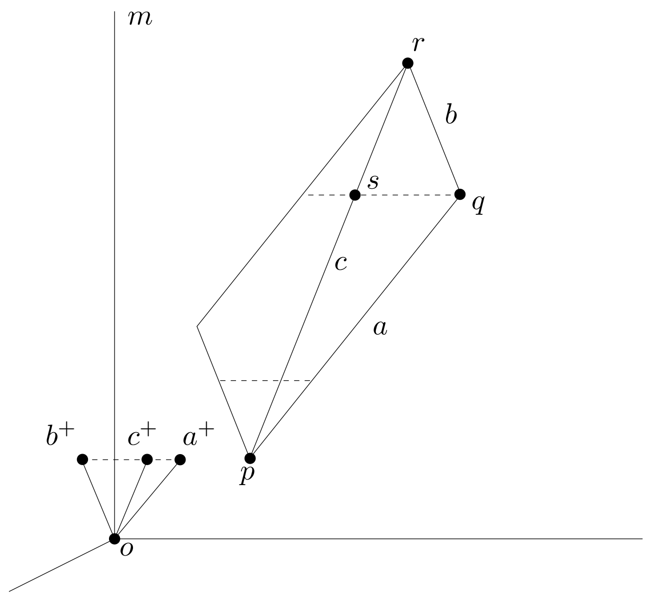

Figure 3.2: for the proof of Lemma 3.1.3.

proof. Let us assume that $m, a, b, c \in Ob$ and $p, q, r \in \mathsf{CD}(m)$ are such that $\mathsf{meetTwp}_m(\widehat{ab}, c)(p, q, r)$ is true. From $\mathsf{Kinem_0}$, it is easy to see that for all observers m, k and points $x, y \in tr_m(k)$ if $m \xrightarrow{\boxdot} k$ then

$$\left|\mathsf{Time}_k^m(x, y)\right| = \frac{|x - y|}{|1_m^k|} \tag{*}$$

holds. Let us abbreviate $(1_m^k)^+$ with k^+ throughout this proof.

First, we show that if c^+ is on the straight-line a^+b^+ then c measures the same time between p, r as a and b do together. Assume that $c^+ \in a^+b^+$. Let s be the intersection of straight-line pr and the straight-line parallel with a^+b^+ through q, see Figure 3.2. Then the triangles oa^+c^+ and pqs are similar; and the triangles oc^+b^+ and rsq are similar, too. Thus

$$\frac{|p - q|}{|1_m^a|} = \frac{|p - s|}{|1_m^c|} \quad \text{and} \quad \frac{|q - r|}{|1_m^b|} = \frac{|s - r|}{|1_m^c|} \tag{**}$$

hold. From this, we have

$$\left|\mathsf{Time}_a^m(p, q)\right| + \left|\mathsf{Time}_b^m(q, r)\right| \overset{*}{=} \frac{|p - q|}{|1_m^a|} + \frac{|q - r|}{|1_m^b|} \overset{**}{=} \frac{|p - s| + |s - r|}{|1_m^c|} = \frac{|r - p|}{|1_m^c|} \overset{*}{=} \left|\mathsf{Time}_c^m(r, p)\right|.$$

Observer c measures more time between p and r iff his time-unit is shorter. Thus we get that $\mathsf{Twp}_m(\widehat{ab} < c)(p, q, r)$ holds iff $\mathsf{convex}(1_m^a, 1_m^c, 1_m^b)$. This completes the proof of the lemma. ■

Now we return to the proof of Theorem 3.1.2. For proving the "$\Longrightarrow$" part, let take three points a', b', c' from MS_m^t such that oa', ob' and oc' are different coplanar lines. From, $\mathsf{AxDisplace}$ there are three observers a, b, c in twin paradox situation such that $1_m^a = a'$, $1_m^b = b'$ and $1_m^c = c'$. Thus from Lemma 3.1.3 we get that MS_m^t is *convex* since a', b', c' were three arbitrary points of it.

The converse is also clear from Lemma 3.1.3 since whenever observers a, b, c are in twin paradox situation, $1_m^a, 1_m^c, 1_m^b$ are on MS_m^t by $\mathsf{AxDisplace}$ as we mentioned before. ■

3.2 Relations between the twin paradox and some symmetry axioms

In this section, we examine the relationship between some symmetry axioms and the twin paradox but before doing so we introduce some definitions.

Simultaneity of observer k at time $t \in F$ observed by observer m is the f^k_m-image of the horizontal hyperplane $\{t\} \times F^{d(k)-1}$:

$$\mathsf{sim}^k_m(t) := f^k_m\big[\{t\} \times F^{d(k)-1}\big] = \big\{p \in \mathsf{CD}(m) : \exists q \in \mathsf{CD}(k)\ \ (p,q) \in f^m_k \wedge q_t = t\big\}$$

p, q are **simultaneous**, in symbols $p \sim^k_m q$, if they are in the same simultaneity for some $t \in F$, i.e. $\exists t \in \bar{t}\ \ p, q \in \mathsf{sim}^k_m(t)$. Notice that if f^k_m is an injective relation, or equivalently f^m_k is a function, then $\sim^k_m$ is an equivalence relation on $Rng(f^k_m)$. Also notice that if $p \in Dom(f^m_k)$ and f^m_k is a function in p, then $p \in \mathsf{sim}^k_m(t)$ iff $f^m_k(p)_t$.

Let $p, q \in F^n$. Then their **Minkowski-product** is $p \bullet q := p_t q_t - p_2 q_2 - \ldots - p_n q_n$. Notice that Minkowski-product is a symmetric bilinear function. We say that p and q are **Minkowski-orthogonal**, in symbols $p \measuredangle q$, iff $p \bullet q = 0$. Two straight-lines, say pq and rs, are Minkowski-orthogonal iff $p - q \measuredangle r - s$. We call a straight-line **self-orthogonal** if it is Minkowski-orthogonal to itself. Notice that $\mathsf{AxPh}^\star$ says that a line is photon-line iff it is a self-orthogonal straight-line or a single point. We call a quadrangle **photon-quadrangle** if its sides are self-orthogonal.

Proposition 3.2.1. The diagonals of a photon quadrangle are Minkowski-orthogonal.

proof. Let p, q, r and s be the vertexes of a photon-quadrangle such as pq, qr, rs and sp are self-orthogonal straight-lines. Then from the definition of the Minkowski-orthogonality we get:

$$\begin{aligned} p \bullet p - 2p \bullet q + q \bullet q &= 0 && (3.1)\\ q \bullet q - 2q \bullet r + r \bullet r &= 0 && (3.2)\\ r \bullet r - 2r \bullet s + s \bullet s &= 0 && (3.3)\\ s \bullet s - 2s \bullet p + p \bullet p &= 0 && (3.4)\end{aligned}$$

from this by simplifying the equitation (3.1) − (3.2) + (3.3) − (3.4) we get:

$$-p \bullet q + q \bullet r - r \bullet s + s \bullet p = 0$$

which is the same as:

$$(r - p) \bullet (q - s) = 0.$$

Thus the diagonals pr and qs of the photon-quadrangle (p, q, r, s) are Minkowski-orthogonal, and this is what we wanted to prove. ■

Let us introduce the following notation for the n-dimensional hyperboloid:

$$Hyp := \{p \in F^n : p_t^2 - p_2^2 - \ldots - p_{n-1}^2 = 1\} = \{p \in F^n : p_t^2 - |p_s|^2 = 1\}.$$

The following axiom is a special case of *Einstein's special principle of relativity* that states that all *inertial* observers are equivalent, cf. Einstein's principle of (general) relativity on page 28, cf. also, e.g., [1, §5] or [6, p.19].

AxSym If two observers strongly see each other, then they see each other's clocks go wrong the same way, formally:

$$\forall m, k \in Ob\ \forall p, q \in \bar{t}\ \ m \overset{\boxdot}{\longleftrightarrow} k \Longrightarrow \mathsf{Time}^k_m(p,q) = \mathsf{Time}^m_k(p,q).$$

This axiom has the following natural weakened form:

AxSymTime If observers m and k strongly see each other then they both see each others time-unit changing the same way:

$$\forall m,k \in Ob \;\; m \overset{\boxdot}{\longleftrightarrow} k \wedge ev_k(o) = ev_m(o) \Longrightarrow (1^k_m)_t = (1^m_k)_t.$$

Let us also introduce an auxiliary axiom here:

Ax⊡ If observer m strongly sees observer k, then k has a brother k' who strongly sees m and well-configured to k:

$$\forall m,k \in Ob \;\; m \overset{\boxdot}{\longrightarrow} k \Longrightarrow \exists k' \in Ob \;\; tr_m(k) = tr_m(k') \wedge ev_k(o) = ev_{k'}(o) \wedge k' \overset{\boxdot}{\longrightarrow} m.$$

The following theorem gives a geometrical characterization for AxSymTime in the models of $\mathsf{Specrel}_0^\star$ and some auxiliary axioms.

Theorem 3.2.2.

$$\mathsf{Specrel}_0^\star \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}, \mathsf{Ax}\boxdot\} \models \mathsf{AxSymTime} \Longleftrightarrow \forall m \in Ob \;\; MS^t_m \subseteq Hyp.$$

proof. In the proof, we will use the following lemma that shows if observer m strongly sees observer k then two points are simultaneous for k iff the straight-line through these points is Minkowski-orthogonal to k's life-line.

Lemma 3.2.3.

$$\mathsf{Specrel}_0^\star \cup \{\mathsf{AxLinTime}\} \models m \overset{\boxdot}{\longrightarrow} k \Longrightarrow [\forall p,q \in Dom(f^m_k) \;\; p \sim^k_m q \iff p - q \measuredangle 1^k_m].$$

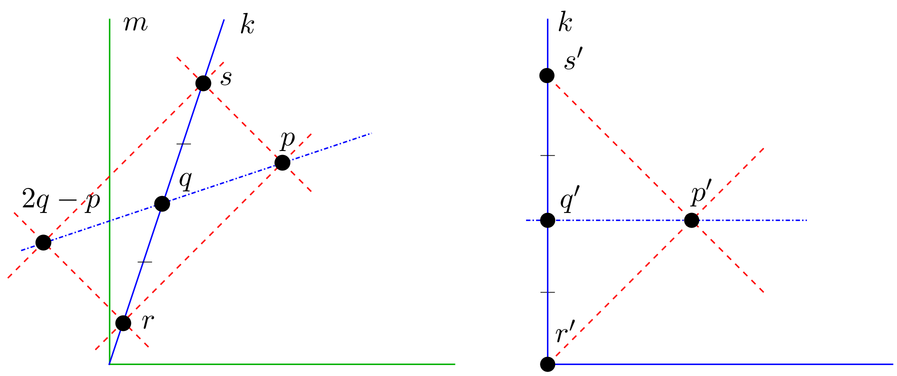

Figure 3.3: for the proof of Lemma 3.2.3.

proof. Let m and k be such observers that m strongly sees k. Let the relation $\backsim \subseteq Dom(f^m_k) \times Dom(f^m_k)$ be defined as $p \backsim q$ holds iff $p - q \measuredangle 1^k_m$. It is easy to see that $\backsim$ is an equivalence relation. Therefore it is enough to prove that it has the same classes as the relation $\sim^k_m$.

Every class of both relations have a unique representative on $tr_m(k)$ because of the following: The relation $\backsim$ has this property because $tr_m(k)$ is a straight-line by Proposition 3.1.1 which is not self-orthogonal by $\mathsf{AxPh}^\star$ and AxSelf; and through every point, there is a Minkowski-orthogonal hyperplane to a non self-orthogonal straight-line which intersect this straight-line in exactly one point. Relation $\sim^k_m$ has a unique representative on $tr_m(k)$ since for all $p \in Dom(f^m_k)$ the $f^k_m\big(f^m_k(p)_t 1_t\big)$ is on $tr_m(k)$ and $\sim^k_m$ equivalent with p.[3]

[3]It is correct to write $f^k_m\big(f^m_k(p)_t 1_t\big)$ since f^k_m is a function in $f^m_k(p)_t 1_t \in \bar{t}$, cf. remark 1.

Let p be an element of $Dom(f_k^m)$ that it is not on the life-line of k. Let r, s be the two points, on k's life-line, that can be reached from p by photon-lines, i.e. by self-orthogonal straight-lines. Let q be the midpoint of r and s, i.e. $q = \frac{r+s}{2}$. See Figure 3.3. We complete the proof by showing that the unique representative of p's class is this q for both relations. For $\backsim$, it is true since $pq \measuredangle rs$ follows from Proposition 3.2.1 since q is the intersection of the diagonals of the photon parallelogram $(p, r, 2q-p, s)$ and r and s are on the life-line of k, cf. Figure 3.3. For proving the statement for $\sim_m^k$, let us denote the f_k^m image of a point x with x'. The statement $q \sim_m^k p$ is true, i.e. $p'_t = q'_t$, because of the following: $p', q', r' \in tr_k(k) = \bar{t}$ since $p, q, r \in tr_m(k)$, q' is the midpoint of r' and s' by AxLinTime, $p'r'$ and $s'p'$ are photon-lines since ps and pr are photon lines. By AxPh*, $r'p'$ and $p's'$ are one-sloped straight-lines. Thus $p'_t = q'_t$. This completes the proof of the lemma. ■

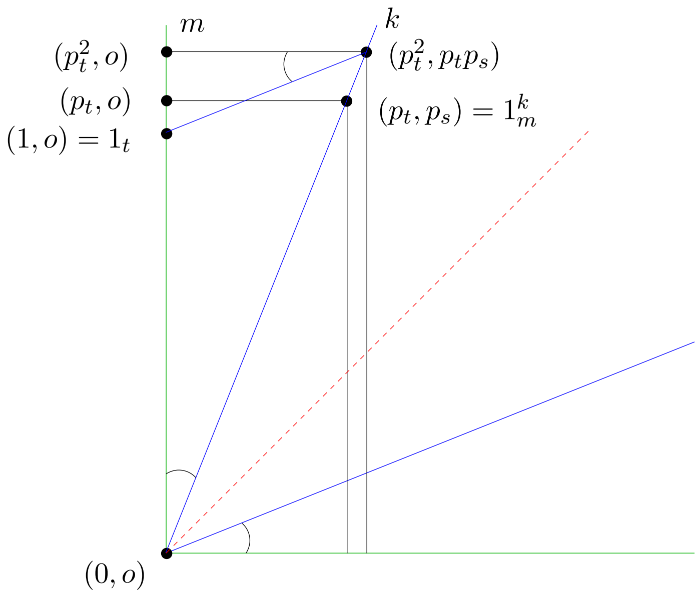

Figure 3.4: for the proof of Theorem 3.2.2.

We return to the proof of Theorem 3.2.2. Let k be a well-configured observer to m, i.e. $ev_k(o) = ev_m(o)$, and let m strongly see k. By Ax$\boxdot$ we can assume that m and k strongly see each other.

If m and k are brothers according to m, i.e. $tr_m(k) = tr_m(m) = \bar{t}$, then AxSymTime is equivalent with $1_m^k = \pm 1_t \in Hyp$.

If m and k are not brothers, then let 1_m^k be the coordinate point $p = (p_t, p_s)$. Thus $(1_m^k)_t = p_t$, cf. Figure 3.4. AxSymTime is true for k and m iff $(1_k^m)_t = (1_m^k)_t$ which is equivalent with the statement $1_t \sim_m^k (p_t^2, p_t p_s) = p_t 1_m^k$ by AxLinTime. From Lemma 3.2.3, we get this is equivalent whit the statement that 1_m^k is Minkowski-orthogonal to $(p_t^2 - 1, p_t p_s)$, i.e. k's life-line $tr_m(k)$ is Minkowski-orthogonal to the the straight-line through 1_t and $(p_t^2, p_t p_s)$. This Minkowski-orthogonality is equivalent that the $(1, o), (p_t^2, p_t p_s), (p_t^2, o)$ and the $(p_t, p_s), (0, o), (p_t, o)$ right-angled triangles are similar since the angles at $(0, o)$ and $(p_t^2, p_t p_s)$ are the same, see for example, [10, p.6]. This similarity is equivalent that $\frac{p_t^2-1}{p_t|p_s|} = \frac{|p_s|}{p_t}$; and this is equivalent with $1 = p_t^2 - |p_s|^2$ which is the same as $1_m^k \in Hyp$.

Since k was an arbitrary observer such that m and k are strongly see each other, this completes the proof of the Theorem. ■

The following corollary of the characterization theorems proved above states that AxSymTime is a stronger assumption than AxTwp in the models of $\mathsf{Specrel}_0^\star$ and some auxiliary axioms:

Corollary 3.2.4. $\mathsf{Specrel}_0^\star \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}, \mathsf{Ax}\boxdot\} \models \mathsf{AxSymTime} \Longrightarrow \mathsf{AxTwp}$.

proof. The statement is clear from Theorem 3.1.2 and 3.2.2 since Hyp is *convex*. ■

Let us introduce the following axiom system of the special relativity:

$$\mathsf{Specrel} := \mathsf{Specrel}_0 \cup \{\mathsf{AxSym}\}.$$

Corollary 3.2.4 has the following trivial but important consequence:

Theorem 3.2.5. $\mathsf{Specrel} \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}\} \models \mathsf{AxTwp}$. ■

The characterization theorems above also imply that $\mathsf{AxSymTime}$ is strictly stronger assumption than AxTwp. Thus it gives a negative answer for the question 4.2.17. in [3].

Corollary 3.2.6. $\mathsf{Specrel}_0 \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}, \mathsf{Ax}\boxdot\} \not\models \mathsf{AxTwp} \Longrightarrow \mathsf{AxSymTime}$.

proof. The statement is clear from Theorems 3.1.2 and 3.2.2 since by the observer-adding construction mentioned above we can easily construct a model for $\mathsf{Specrel}_0 \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}, \mathsf{Ax}\boxdot\}$ where MS^t_m is *convex* but not a subset of Hyp. ■

There is an another phenomenon in relativity about moving clocks. We formulate the named phenomenon with the following axiom:

$\mathsf{AxSlowTime}$ Every observer sees every (relatively) moving observer's clocks slowing down, formally:

$$\forall m, k \in Ob \;\; m \xrightarrow{\boxdot} k \wedge tr_m(k) \neq tr_m(m) \Longrightarrow \left|(1^k_m)_t\right| > 1.$$

This phenomenon also has strong connection with the twin paradox.

We call the less-than-one sloped straight-lines slower-than-light STL straight-lines. We will generalize this notation on page 26 below. The following axiom states that the uniform motion is possible for the observers with any speed less than the speed of the light.

AxOb Every STL straight-line is a trace of some observer:

$$\forall m \in Ob \;\; \{tr_m(k) : k \in Ob\} \supseteq \{l \in \mathsf{Lines}_{d(m)} : \mathsf{slope}(l) < 1\}.$$

The following theorem gives a geometrical characterization for $\mathsf{AxSlowTime}$ in the case when F is the ordered field of the reals.

Theorem 3.2.7. If we assume that $F = \mathbb{R}$, then

$$\mathsf{Specrel}_0 \cup \{\mathsf{AxLinTime}, \mathsf{AxOb}, \mathsf{AxDisplace}\} \models \mathsf{AxSlowTime} \iff MS^t_m = Hyp.$$

sketch of the proof. The "$\Longleftarrow$" part is trivial. Let S be an arbitrary plane which contains the time-axis. We prove the "$\Longrightarrow$" part by proving that the intersection of S with the Minkowski-sphere is the same as with the $d(m)$ dimensional hyperboloid for every observer m.

For every point p, there is a unique hyperplane which is Minkowski-orthogonal to op. This hyperplane determines two half-spaces; let us call the upper half-space **positive Minkowski half-space**. $\mathsf{AxSlowTime}$ is equivalent with the statement $(MS^t_m)^+$ is the subset of the positive Minkowski half-spaces which determined by the time-unit vector 1_t. From this and Lemma 3.2.3 we get that $\mathsf{AxSlowTime}$ is equivalent with the statement that $(MS^t_m)^+$ is a subset each of the Minkowski half-spaces determined by the points of $(MS^t_m)^+$.

From AxOb, we get that the intersection of an STL straight-line with $(MS^t_m)^+$ is one and only one point. We associate a function to the intersection of S with $(MS^t_m)^+$ from the open interval $(-1,1) \subseteq F$ to F^+ by the following definition: if $x \in (-1,1)$, then $MS^t_m(x)$ is the time part of the intersection of the STL straight-line through o and $(1,x) \in S$ and $(MS^t_m)^+$.

We associate a function to the intersection of S and the set Hyp in the same way. From the definitions, it is not hard to see that both $MS^t_m(x)$ and $Hyp(x)$ are differentiable functions whose differentials are the same. Since F is the ordered field of the reals, $MS^t_m(x)$ and $Hyp(x)$ can only differ in a constant but $Hyp(0) = 1_t = MS^t_m(0)$. Thus $MS^t_m(x) = Hyp(x)$ as we wanted. ■

3.3 Twin paradox and the possibility of faster-than-light motion for observers

In this section, we show that if the twin paradox and some auxiliary axioms are added to $\mathsf{Specrel}_0$ then the faster-than-light motion is not allowed for the observers in its models.

The following theorem states that if the twin paradox axiom is true, then there is no faster-than-light observer in the models of $\mathsf{Specrel}_0$ and some auxiliary axioms:

Theorem 3.3.1. $\mathsf{Specrel}_0 \cup \{\mathsf{AxLinTime}, \mathsf{AxOb}\} \models \mathsf{AxTwp} \Longrightarrow \mathsf{noFTL}$.

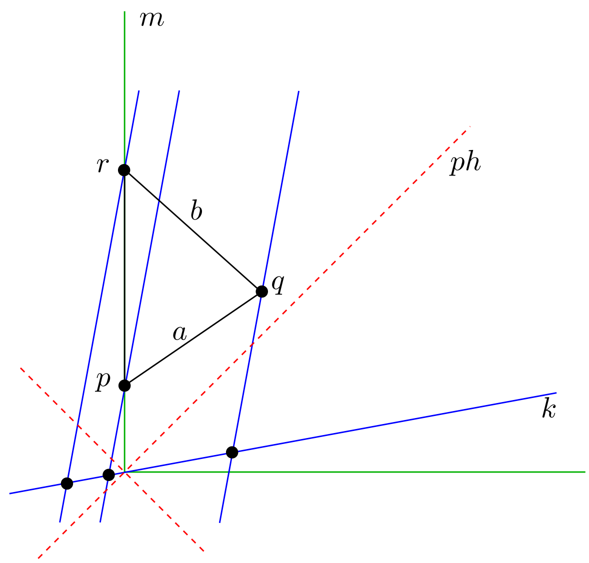

Figure 3.5: for the proof of the Theorem 3.3.1.

proof. If there is an FTL observer, say k, then from AxOb there are some observers a and b such that $\mathsf{meetTwp}_m(\widehat{ab}, m)(p,q,r)$ and $\mathsf{meetTwp}_k(\widehat{mb}, a)(p,r,q)$, cf. Figure 3.5. Thus from AxTwp we get $\mathsf{Twp}_m(\widehat{ab} < m)(p,q,r)$ and $\mathsf{Twp}_k(\widehat{mb} < a)\big(f^m_k(p), f^m_k(r), f^m_k(q)\big)$. This means that m measured more time in the section $[pr]$ than a measured in the section $[pq]$ plus b measured in the section $[qr]$ and a measured more time in the section $[pq]$ than m measured in the section $[pr]$ plus b measured in the section $[rq]$. Thus m measures more time in the section $[pr]$ than himself, and this is impossible.[4] ■

[4]Notice that this proof did not use the whole power of the axiom AxOb.

3.4 Weakened forms of the twin paradox axiom

In this section, we check what happens if we change the quantifiers in the twin paradox axiom; and we also turn our attention toward what kind of axioms are experimentally testable in some sense.

If we change the quantifiers in the AxTwp axiom, we get the following ones:

$\mathsf{Ax\forall\exists Twp}$ All observers see twin paradox in some twin paradox situations:[5]

$$\forall m \in Ob\ \exists a,b,c \in Ob\ \exists p,q,r \in \mathsf{CD}(m)\ \ \mathsf{meetTwp}_m(\widehat{ab},c)(p,q,r) \wedge \mathsf{Twp}_m(\widehat{ab} < c)(p,q,r).$$

$\mathsf{Ax\exists\forall Twp}$ There is some observer who sees twin paradox in all the twin paradox situations:

$$\exists m \in Ob\ \ \forall a,b,c \in Ob\ \ \forall p,q,r \in \mathsf{CD}(m)$$
$$\mathsf{meetTwp}_m(\widehat{ab},c)(p,q,r) \Longrightarrow \mathsf{Twp}_m(\widehat{ab} < c)(p,q,r).$$

$\mathsf{Ax\exists\exists Twp}$ There is some observer who sees twin paradox in some twin paradox situations:

$$\exists m \in Ob\ \exists a,b,c \in Ob\ \exists p,q,r \in \mathsf{CD}(m)\ \ \mathsf{meetTwp}_m(\widehat{ab},c)(p,q,r) \wedge \mathsf{Twp}_m(\widehat{ab} < c)(p,q,r).$$

Notice that these axioms have the hierarchy shown in Figure 3.6 since the class of models of $\mathsf{Specrel}_0^\star$ closed under taking disjoint union and since in some of the models the twin paradox is true while in others it is not.

The following corollary shows that $\mathsf{Ax\forall\exists Twp}$ is strictly weaker assumption than the twin paradox; thus it answers the question 4.2.16 in [3].

Corollary 3.4.1. $\mathsf{Specrel}_0 \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}\} \not\models \mathsf{Ax\forall\exists Twp} \Longrightarrow \mathsf{AxTwp}$.

proof. The statement is clear from Lemma 3.1.3 and Theorem 3.1.2 since by the observer-adding construction mentioned above we can easily construct a model for the axiom system $\mathsf{Specrel}_0 \cup \{\mathsf{AxLinTime}, \mathsf{AxDisplace}\}$ where there are points $p,q,r \in MS_m^t$ such that $\mathsf{convex}(p,q,r)$ but MS_m^t is not *convex*. ■

We can interpret Corollaries 3.4.1 and 3.2.6 to says that we cannot prove that $\mathsf{Specrel}$ is "true" in the "physical world" by checking the twin paradox, not even if we assume $\mathsf{Specrel}_0$ and checking all the possible twin paradox situations.

We can associate a naive physical meaning to existential formulas, i.e. formulas that contains only $\exists$ quantifier, e.g., $\mathsf{Ax\exists\exists Twp}$. We can say that they are experimentally testable.

A natural question arises here: is $\mathsf{Specrel}$ experimentally testable over some natural axioms of kinematics? By this question we mean the following: are there some (maybe infinitely many) existential axioms and some natural consequences of $\mathsf{Specrel}_0$ not talking about photons (containing any kind of quantifiers) whose models are the same as the models of $\mathsf{Specrel}$? Like in the situation in geometry where there is an existential axiom, i.e. the axiom saying that there is a triangle such that sum of its angles is π, that lifts the Euclidean plane out of the models of the plane geometries while the standard axiom of parallelism is not an existential one.

Theorem 3.4.2 shows that the answer for this question is negative. Moreover, $\mathsf{Specrel}$ is not experimentally testable from $\mathsf{Specrel}_0$.

Theorem 3.4.2. There is no set of existential formulas which together with $\mathsf{Specrel}_0$ have the same models as $\mathsf{Specrel}$.[6]

[5]Since this (the inertial) formulation of the twin paradox loses its meaning when we change some of the quantifiers to an existential one, we change the axioms to mean what they would were they the accelerated versions.

[6]The statement remains true if we change $\mathsf{Specrel}_0$ in it to any set of consequences of $\mathsf{Specrel}_0$.

$\{\mathsf{AxLinTime}, \mathsf{AxSelf}, \mathsf{AxEvTr}, \mathsf{AxDisplace}\} \models$

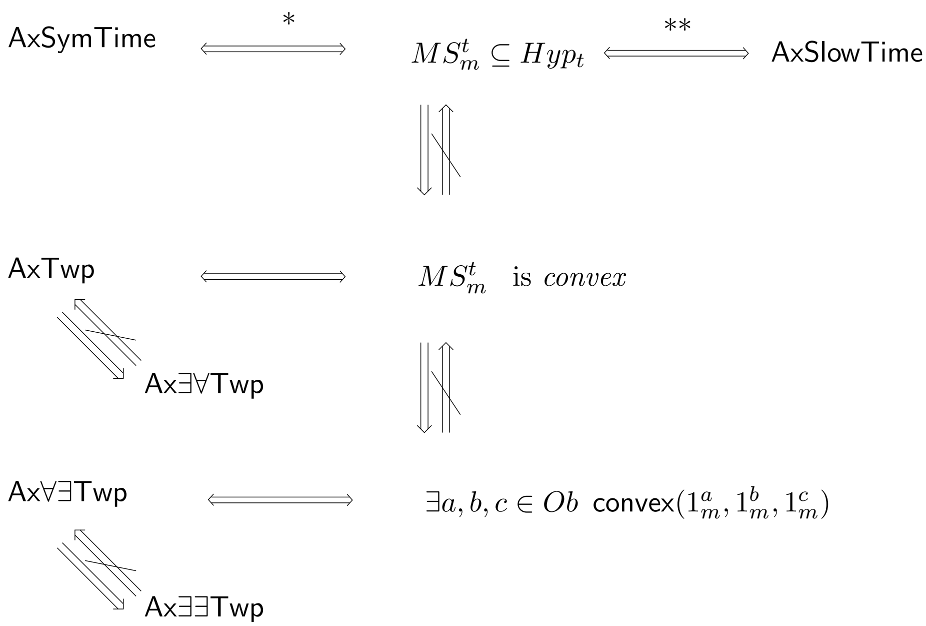

Figure 3.6: $* = \{\mathsf{AxPh}^\star, \mathsf{Ax}\boxdot\}$ $** = \{\mathsf{AxPh}^\star, \mathsf{AxEv}, \mathsf{AxOb}, \mathsf{F} = \mathbb{R}\}$. Recall that $\mathsf{Kinem}_0 = \{\mathsf{AxLinTime}, \mathsf{AxEvTr}, \mathsf{AxSelf}\}$ and $\mathsf{Specrel}^\star_0 = \{\mathsf{AxSelf}, \mathsf{AxPh}^\star, \mathsf{AxEvTr}\}$.

proof. It is easy to see that if some theory Γ is axiomatizable in an other theory Σ by only existential formulas, then it must be preserved the extensions cf., e.g., [17]. It is clear from the observers adding construction and Theorem 3.2.2 that we can extend a Specrel model to a $\mathsf{Specrel}_0$ model which is not a Specrel model. ■

Chapter 4

Accelerated observers

In this chapter, we extend the range of our interest toward accelerated observers and introduce some axiom systems that permit their existence. We generalize the theorems and definitions of Real Analysis that we use in this chapter for arbitrary ordered fields in the Appendix.

4.1 On the possibility of non-inertial motion in models of $\mathsf{Specrel}_0$

In this section, we show that $\mathsf{Specrel}_0$ permits acceleration only in its two-dimensional models. We also draw up a conjecture about the existence of such models of $\mathsf{Specrel}_0$ where acceleration is possible but only in a relative sense, cf. Einstein's principle on page 28.

Let us recall the following well known theorem from the literature:

Theorem 4.1.1 (Alexandrov-Zeeman). Let $n > 2$. If $f : F^n \longrightarrow F^n$ is a bijection that maps one-sloped straight-lines onto one-sloped straight-lines, then f maps any straight-line onto a straight-line, i.e. f is a collineation.[1]

From Theorem 4.1.1, it follows that the trace of an observer has to be straight-line in the more-than-two dimensional models of $\mathsf{Specrel}_0$ since it is the image of the time-axis by the world-view transformation which is a collineation. Therefore we cannot examine accelerated motion of observers in these models; but we would like to talk about accelerated observers, too. Thus we have to change our axiom system but before doing so; let us see what is true in the two-dimensional models of $\mathsf{Specrel}_0$.[2]

We define the **life-curve** $Tr_m^k : F \longrightarrow \mathsf{CD}(m)$ of observer k as seen by observer m as $t \mapsto Tr_m^k(t) := f_m^k(t, o)$.

Proposition 4.1.2. There is a two-dimensional model of $\mathsf{Specrel}_0$ where the curve $\gamma : F \longrightarrow F^2$ is the life-curve of some observer iff $Rng(\gamma)$ intersects every photon-line once and only once.

proof. It is clear that every life-curve has the named property in the two-dimensional models of $\mathsf{Specrel}_0$ since an observer cannot meet a photon more than once but has to meet every photon.

Let m be an observer in a two-dimensional model of $\mathsf{Specrel}_0$ and let $\gamma : F \longrightarrow F^2 = \mathsf{CD}(m)$ be a curve which has the named property. We are going to extend this model with a new observer k such that $Tr_m^k = \gamma$ will hold.

[1] We call a transformation **collineation** if it takes straight-lines to straight-lines.

[2] We will change our axiom system to accommodate accelerated observers in §4.2.

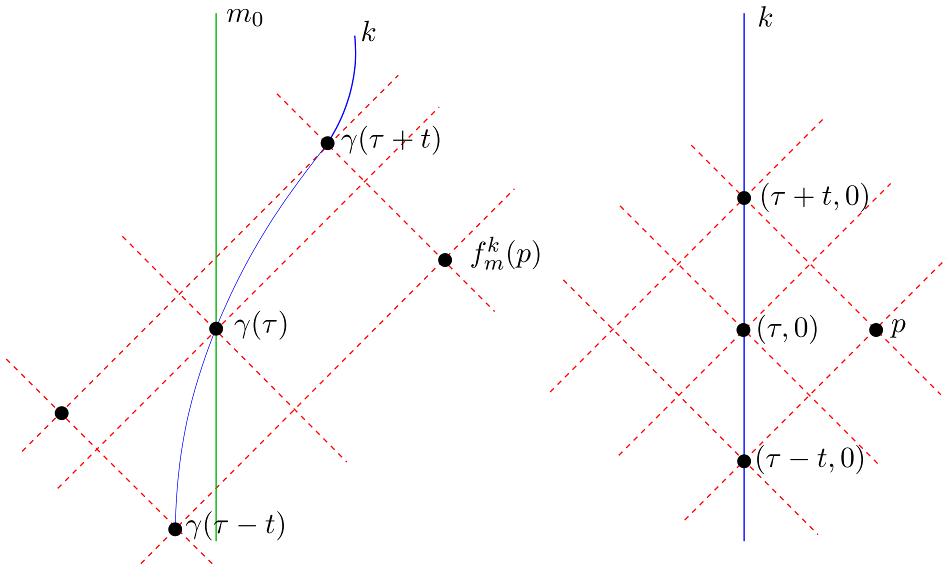

Figure 4.1: for the proof of the Proposition 4.1.2.

First, we build the world-view transformation $f_m^k : \mathsf{CD}(k) = F^2 \longrightarrow F^2 = \mathsf{CD}(m)$ from γ. Let $f_m^k((\tau,0)) := \gamma(\tau)$ for each $\tau \in F$. This step will make AxSelf valid for k. For a point p which is not in the time-axis, let $(\tau+t,0)$ and $(\tau-t,0)$ be the points where the photon-lines through p intersect the time-axis cf. Figure 4.1. There are two intersections of the photon-lines through $\gamma(\tau+t)$ and $\gamma(\tau-t)$. Let us choose on the basis of these two points $f_m^k(p)$ such that f_m^k will take parallel photon-lines to parallel photon-lines. We have freedom in choosing f_m^k just for the first point. This step will make AxPh valid and will not ruin the validity of AxSelf. Let we change the even function of m such that we put k into all the events that m sees in the points of $Rng(\gamma)$. Then we change the other observers event function via the world-view transformations, cf. the radar construction on page 35. AxEv also becomes valid after this last step since γ intersects all the photon-lines. ■

Notice that the curve property in Proposition 4.1.2 is independent from the parametrization. This means that we can rearrange some observer's inner time via an arbitrary permutation.

The following Corollary shows that a life-line of an observer can be very different from a straight-line in the two-dimensional models of $\mathsf{Specrel}_0$.

Corollary 4.1.3. There is a two-dimensional model of $\mathsf{Specrel}_0$ where the life-line of an observer is dense.[3]

outlined proof. It is enough to show that there is a dense subset of F^2 which intersects every photon-line once and only once. If a set intersects the interior of every photon-square, then it is dense. Since photon-squares have the same cardinality as the photon-lines, the named set can be easily constructed by transfinite induction. ■

Let $p \in F^n$. We use the following notation: $\Lambda_p := \{q \in F^n : \mathsf{slope}(p-q) = 1\}$ for the **light-cone** (through p). $\Lambda_p^+ := \{q \in F^n : \mathsf{slope}(p-q) = 1 \wedge q_t > p_t\}$ for the **future light-cone**. $\Lambda_p^- := \{q \in F^n : \mathsf{slope}(p-q) = 1 \wedge q_t < p_t\}$ for the **past light-cone**. $I^-(p) := \{q \in F^n : \mathsf{slope}(p-q) < 1 \wedge q_t < p_t\}$ is called **chronological past**. $I^+(p) := \{q \in F^n : \mathsf{slope}(p-q) < 1 \wedge q_t > p_t\}$ is

[3] A subset of F^n is called **dense** if it intersects every non empty open set.

called **chronological future**. For these notion, cf., e.g., [11, §6]. Two distinct points, p and q are **time-like separated** iff they are on a straight-line of slope less-than-one. They are **light-like separated** iff they are on a straight-line of slope one. They are **space-like separated** iff they are on a straight-line of slope more-than-one. We say that a set $H \subset F^n$ is **slower-than-light** (STL) iff any two distinct elements of it are time-like separated. A set $H \subset F^n$ is **faster-than-light** (FTL) iff any two distinct elements of it are space-like separated. A curve $\gamma : F \longrightarrow F^n$ is $\mathsf{STL}/\mathsf{FTL}$ iff $Rng(\gamma)$ is $\mathsf{STL}/\mathsf{FTL}$. We say that a body b is $\mathsf{STL}/\mathsf{FTL}$ according to observer m (in symbols $b\ \mathsf{STL}\ m$ / $b\ \mathsf{FTL}\ m$) if its trace is $\mathsf{STL}/\mathsf{FTL}$ in $\mathsf{CD}(m)$. Also notice that most of the curves are neither STL nor FTL. We call p **time-like vector** iff its slope is less-than-one. We call p **light-like vector** iff its slope equals one. We call p **causal vector** iff its slope is less or equal to one. Let $\gamma : F \to F^n$ be a curve. The curve γ is called **time-like** if it is differentiable and its derivate is a time-like vector in each point, i.e. $\forall t \in F \quad \mathsf{solpe}(\gamma'(t)) < 1$. The curve γ is called **causal** if if it is differentiable and its derivate is a causal vector in each point, i.e. $\forall t \in F \quad \mathsf{solpe}(\gamma'(t)) \leq 1$. We call p **future-directed** vector if $p_t > 0$. We say that a time-like curve is **well-parametrized** iff its derivate, in every point, is a future-directed vector that has one Minkowski-lenght, i.e. $\forall t \in F \quad \mu(\gamma'(t)) = 1 \wedge \gamma'(t)_t > 0$. By a **chord** of γ we mean a straight-line passing through to two distinct points of $Rng(\gamma)$. The following theorem shows how the (differentiable) time-like, STL and causal curves are related:

Theorem 4.1.4. Let $\gamma : F \longrightarrow F^n$ be a curve. Then

(i) γ is time-like $\overset{F=\mathbb{R}}{\Longrightarrow}$ γ is STL $\Longrightarrow$ γ is causal.[4]

(ii) γ is causal $\not\Rightarrow$ γ is STL $\not\Rightarrow$ γ is time-like.

proof. Assume γ is not STL and $F = \mathbb{R}$.[5] Then it has a light-like or a space-like chord. Let H be a hyperplane that contains this chord and does not contain time-like straight-lines. Let $\pi_H : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the parallel projection to $\bar{t}$ w.r.t. H, i.e. $\pi_H(p) = t$ iff $p - (t, o)$ is parallel to H. Thus by applying the Rolle's Theorem to $\gamma \circ \pi_H$ we get that there is $c \in F$ such that $(\gamma \circ \pi_H)'(c) = 0$. Thus $\gamma'(c)$ is parallel with H since $(\gamma \circ \pi_H)'(c) = \pi_H(\gamma'(c))$ by Corollary A.0.8. Since H does not contain time-like straight-lines, $\gamma'(t)$ is not time-like. Thus γ is not time-like.

If γ is an STL curve, then every chord of it is time-like therefore its derivate is a causal vector in each point. Thus γ is a causal curve.

Any self-orthogonal line can be parametrized to be a causal curve but it cannot be the range of an STL curve.

It is easy to construct a curve that has causal derivate vectors but every chord of it is time-like, e.g., $F \ni t \mapsto (t, \frac{1}{3}t^3 + t) \in F^2$ is good, cf. Figure 4.2. ∎

Conjecture 2. Assume γ is (continuously) differentiable and $F = \mathbb{R}$. γ is STL iff γ is causal and the set $\{t \in F : \gamma'(t)$ is light-like$\}$ does not contain intervals.

In F^n, an STL set intersects every horizontal (orthogonal to the time-axis) hyperplane at most once, and an FTL set intersects every vertical (parallel with the time-axis) straight-line at most once. Therefore we can associate a function from a subset of F to F^{n-1} to every STL set and a function from a subset of F^{n-1} to F to every FTL set. We call these functions **associated functions**.

Proposition 4.1.5. The associated function to an $\mathsf{STL}/\mathsf{FTL}$ set is uniformly continuous.

[4] The condition $F = \mathbb{R}$ can be changed with a first order axiom scheme, cf. **(???)**

[5] If $F \neq \mathbb{R}$ then there are non STL time-like curves, e.g., if $\emptyset \neq H \subset F$ is a closed and open set then the set $\{(t, 0) \in F^2 : t \in H\} \cup \{(t, 1) \in F^2 : t \notin H\}$ can be parametrized to be a good counterexample.

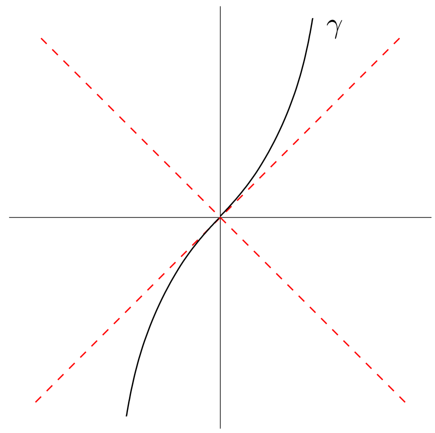

Figure 4.2: to the proof of Theorem 4.1.4.

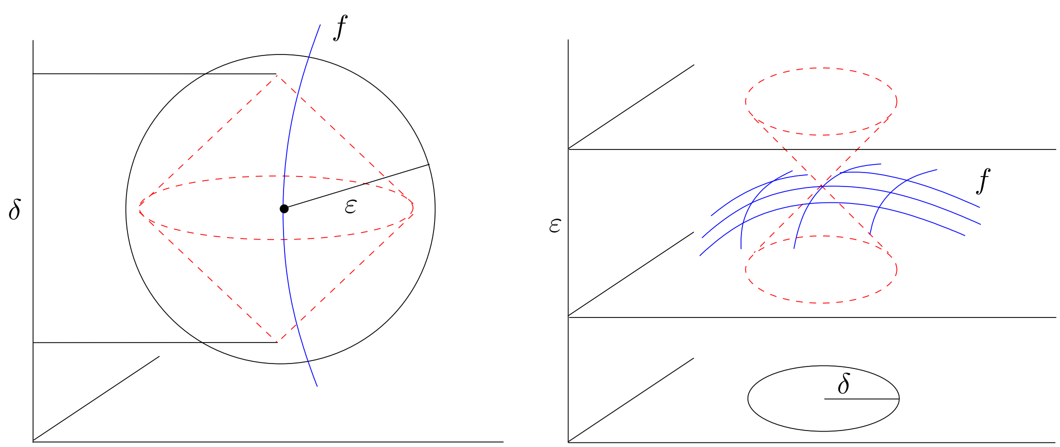

Figure 4.3: to the proof of Proposition 4.1.5.

on the proof. The proof can be read from Figure 4.3. Moreover, the associated functions f have the property $|f(x) - f(y)| < |x - y|$ and this is also clear from the definitions. ■

We say that body b is **eternal** for observer m (in symbols b ETR m) if b is present in every simultaneity of m, formally: $\forall t \in \bar{t}\ \ tr_m(b) \cap \mathsf{sim}_m^m(t) \neq \emptyset$. We also say that body b is **everseen** for observer m (in symbols b EVS m) if the life-line of b intersects all of the future and past light-cones through the points of the time-axis, formally: $\forall t \in \bar{t}\ \ tr_m(b) \cap \Lambda^-(t) \neq \emptyset \wedge tr_m(b) \cap \Lambda^+(t) \neq \emptyset$.

In physics, *Einstein's principle of (general) relativity* (PR) says that *all observers* are equivalent from the point of view of physics, cf., e.g., [1, §5] or [6, p.130]. We get a stronger version of PR if we change from the physical point of view to the point of view of the formal logic, cf. [14, §2.8.3].[6] From this point of view, we can say that the strong version of PR is true in a frame model if the automorphism group acts transitively on the Ob_n relations, i.e. iff for every two (same dimensional) observers m and k there is some automorphism $\psi \in Aut(\mathfrak{M})$ of the frame model $\mathfrak{M}$ such that $\psi(m) = k$, cf. [14, Theorem 2.8.20]. The following conjecture is about the existence of some two-dimensional frame models of $\mathsf{Specrel}_0$ where this strong version of PR holds but non-inertial motions are allowed for the observers. Notice that in the higher dimensional models of $\mathsf{Specrel}_0$ this is impossible, cf. the Alexandrov-Zeeman theorem on page 24.

CONJECTURE 3. There is a two-dimensional model $\mathfrak{M}$ of $\mathsf{Specrel}_0$ where "the strong version of Einstein's principle of relativity" is true and $\exists m, k \in Ob\ \ tr_m(k) \notin \mathsf{Lines}$.

The idea of the conjectured proof is the following: If we add an observer to every (or every continuous) EVS slower-than-light curve by the construction used in Propositions 4.1.2 in both possibilities, then we get the desired model.

4.2 The accelerating axiom

In this section, we extend $\mathsf{Specrel}$ to $\mathsf{Specrel}^+$ where the acceleration will be allowed for observers and introduce the axiom AxAcc that will give a connection between the coordinate systems of the accelerated observers and of the inertial observers.

We say that k, m are **co-moving observers** at $q \in F^n$, in symbols $k \asymp_q m$, if (1)-(3) below hold:

(1) $ev_m(q) = ev_k(q) \supseteq \{k, m\}$

(2) f_k^m and f_m^k are injective functions on $B_\varepsilon(q)$ for some $\varepsilon \in F^+$, and

(3) $\forall \varepsilon \in F^+\ \exists \delta \in F^+\ \ \forall p \in B_\delta(q)\ \ \big|p - f_k^m(p)\big| \leq \varepsilon\big|p - q\big|$ and $\big|p - f_m^k(p)\big| \leq \varepsilon\big|p - q\big|$.

Notice that co-moving observers have to have the same dimensions. Behind the definition of the co-moving observers is the following intuitive image: as we zoom into smaller and smaller neighborhoods of a given coordinate point the world-views of two observers are more and more the same.

Also notice that there are some redundancies in the definition of the co-moving observers. $ev_m(q) = ev_k(q)$ immediately follows from (3) if we choose p and q to be the same. It is also easy to prove that (2) follows from (3) by Convention 3.

If the restriction of the world-view transformation to some sphere is a function, then we can talk about its differentiability.[7]

[6] The footnotes of the works of H. Andréka, J. X. Madarász and I. Németi very often contain *important informations*, so *do not omit them.*

[7] If a restriction of a relation is a function, we call this relation differentiable if the restriction is differentiable and the differential of this relation is the differential of the restriction.

(3.1) f_m^k and f_k^m are differentiable in q and their differentials are the identity map.

Proposition 4.2.1. (1), (2), (3) are true for two observers iff (1), (2), (3.1) are true for them.

proof. The proof of the statement is straightforward from the definitions. ■

Proposition 4.2.2. The co-moving relation $\asymp_q$ is an equivalence relation on $Dom(\asymp_q)$, i.e. on the set $\{k \in Ob : \exists m \in Ob \quad m \asymp_q k\}$.

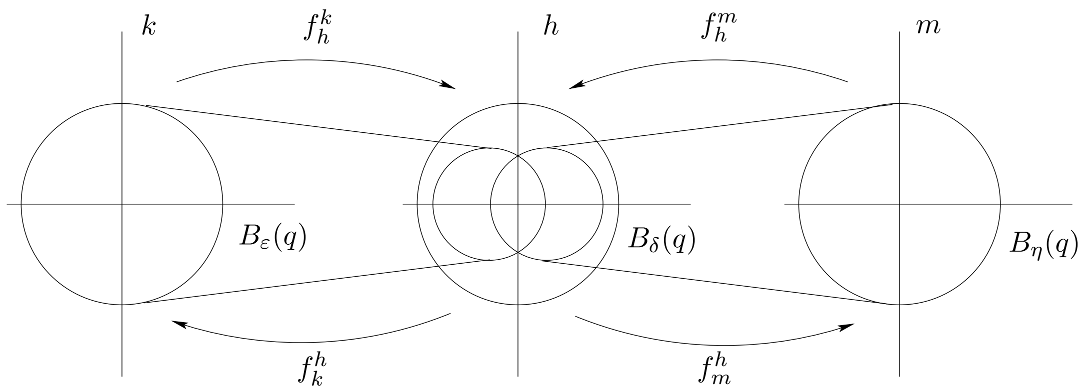

Figure 4.4: for the proof of Proposition 4.2.2.

proof. The symmetry of $\asymp_q$ is trivial. Assume $m \asymp_q h$ and $h \asymp_q k$. There is some δ such that f_m^h and f_k^h are injective functions on $B_\delta(q)$. From differentiability of f_h^k and f_h^m follows their continuity. Therefore there is some η such that $f_h^m[B_\eta(q)] \subseteq B_\delta(q)$ and f_h^m is an injective function on $B_\eta(q)$. Similarly there is some ε such that $f_m^k[B_\varepsilon(q)] \subseteq B_\delta(q)$ and f_h^k is an injective function on $B_\varepsilon(q)$ cf. Figure 4.4. For these ε, δ and η the following holds: $f_m^k\big|_{B_\varepsilon(q)} = f_h^k\big|_{B_\delta(q)} \circ f_m^h\big|_{B_\eta(q)}$, since $ev_h\big|_{B_\delta(q)} \subseteq ev_k\big|_{B_\varepsilon(q)} \cap ev_m\big|_{B_\eta(q)}$, cf. remarks on page 7 below convention 3. From this, the transitivity can be easily derived since composition of injective and differentiable functions is injective and differentiable and since the differential of the composition of two functions is the composition of their differentials, cf. the chain rule on page 39.

The reflexivity follows from the symmetry and the transitivity of an arbitrary relation on the domain. $\big($moreover $f_k^k\big|_{B_\varepsilon(q)} = Id\big|_{B_\varepsilon(q)}$ is also true for some ε if $k \in Dom(\asymp_q)\big)$. ■

We introduce a new unary relation symbol Ib for **inertial bodes**. The observers in the set $IOb := Ob \cap Ib$ are called **inertial observers**. If Ax is an axiom, then $\mathsf{Ax}^{\mathsf{in}}$ will denote the axiom we get by restricting it to the inertial observers, i.e. changing all occurrences of Ob_n to IOb_n.

We introduce the following axiom to exclude the undesirable two-dimensional models of $\mathsf{Specrel_0}$, cf. Proposition 4.1.2.

AxLine Traces of the observers are straight-lines

$$\forall m, k \in Ob \;\; tr_m(k) \in \mathsf{Lines}_{d(m)}.$$

We also introduce the following weakened form of AxSym:

$\mathsf{AxSym_0}$ If two observers strongly see each other, then they see each other's clocks go wrong the same way in absolute value, formally:

$$\forall m, k \in Ob \;\; \forall p, q \in \bar{t} \;\; m \overset{\boxdot}{\longleftrightarrow} k \Longrightarrow \big|\mathsf{Time}_m^k(p, q)\big| = \big|\mathsf{Time}_k^m(p, q)\big|.$$

The following axiom is a very convenient technical axiom:

AxExt If two observers' event function is the same, then they are the same observers.

$$\forall m, k \in Ob \;\; ev_m = ev_k \Longrightarrow m = k.$$

Let us introduce the following axiom system as the first step towards generalizing Specrel for being able to discuss accelerated observers:

$$\mathsf{Specrel}^+ := \{\mathsf{AxSelf}, \mathsf{AxExt}, \mathsf{AxPh}^{\mathsf{in}}, \mathsf{AxEv}^{\mathsf{in}}, \mathsf{AxSym}_0^{\mathsf{in}}, \mathsf{AxLine}^{\mathsf{in}}, \mathsf{AxOb}^{\mathsf{in}}\}.$$

We introduce the promised axiom that connects the coordinate systems of the accelerated and of the inertial observers:

AxAcc At any point on the life-line of any observer there is a co-moving inertial observer:

$$\forall k \in Ob \;\; \forall q \in tr_k(k) \; \exists m \in IOb \;\; m \asymp_q k.$$

Notice that, form $\mathsf{Specrel}^+$, it follows that two co-moving inertial observer have to be the same, by AxExt, Theorem 4.2.3 and Proposition 4.2.2. Thus, form AxAcc and AxSelf, we get that for all $k \in Ob$ and there is an unique inertial co-moving observer. For this unique observer we use the notation k_t and we call it **the inertial co-moving observer** of k at t.

In an arbitrary ordered field, the basic tools of the Real Analysis are not present. For example the Bolzano theorem, which says that the continuous image of an interval is an interval, is not true in any other ordered field than the field of the real numbers. Since $\mathbb{R}$ is the only connected ordered field, the characteristic function of a closed and open set is a good counterexample in the other ordered fields. To replace the missing tools, we will introduce an axiom scheme called IND, see below.

Let φ be a first-order formula in a language L that contains a binary relation symbol $\leq$ and a unary relation symbol F; and let $t, a_1, \ldots, a_n$ be all the free variables of φ. We use the following abbreviation for stating that $b \in F$ is a bound of the set defined by the formula φ on F when using $a_1, \ldots a_n$ as fixed parameters:

$$\mathsf{bound}_\varphi(b) :\Longleftrightarrow \forall t \in F \;\; \varphi(t) \Longrightarrow t \leq b.$$

AxSup_φ The set $\{t \in F : \varphi(t, a_1, \ldots, a_n)\}$ defined by φ when using $a_1 \ldots a_n$ as fixed parameters has a supremum if it is nonempty and bounded i.e.:

$$\forall a_1, \ldots, a_n \;\; [\exists t \in F \;\; \varphi(t)] \wedge [\exists b \in F \;\; \mathsf{bound}_\varphi(b)] \Longrightarrow$$
$$[\exists d \in F \;\; \forall c \in F \;\; \mathsf{bound}_\varphi(c) \Longrightarrow d \leq c \wedge \mathsf{bound}_\varphi(d)].$$

Let us introduce the following axiom scheme for filling the undesired gaps in the fields different from $\mathbb{R}$:

$$\mathsf{IND}_L := \{\mathsf{AxSup}_\varphi : \varphi \text{ is a first-order formula in the language } L\}.$$

We omit the subscript L of the IND_L if L is our frame language.

Let us introduce the following two axiom systems for the accelerated version of the special relativity theory:

$$\mathsf{AccRel}_0 := \mathsf{Specrel}^+ \cup \{\mathsf{AxAcc}, \mathsf{AxEvTr}\}.$$

$$\mathsf{AccRel} := \mathsf{Specrel}^{+} \cup \{\mathsf{AxAcc}, \mathsf{AxEvTr}, \mathsf{IND}\}.$$

Notice that AccRel is an extension of $\mathsf{Specrel}^{+}$, i.e. if $Ob = IOb$ in a model of AccRel then it is a model of $\mathsf{Specrel}^{+}$.

Let

$$\mu(p) := \begin{cases} \sqrt{\left|p_t^2 - |p_s|^2\right|} & \text{if } p_t^2 - |p_s|^2 \geq 0 \\ -\sqrt{\left|p_t^2 - |p_s|^2\right|} & \text{otherwise} \end{cases}$$

be the **Minkowski-length** of $p \in F^n$ and let the **Minkowski-distance** between p and q be $\mu(p,q) := \mu(p-q)$. We call a transformation **Poincaré-transformation** if it preserves the Minkowski-distance.

Let $\rho : F^2 \to F^2$ denote "the reflection to the $x = y$-line", i.e. $\rho(x,y) = (y,x)$ for all $x, y \in F$. Notice that ρ is a linear transformation which does not preserve the Minkowski-length but preserves the absolute value of the Minkowski-length since $\mu(\rho(p)) = -\mu(p)$. We will use the following theorem from the literature:

Theorem 4.2.3. Assume $\mathsf{Specrel}^{+}$. Let m, k be inertial observers. Then (i) and (ii) below hold:

(i) f_k^m is a Poincaré-transformation if $n > 2$.

(ii) f_k^m is a Poincaré-transformation composed perhaps with the reflection ρ if $n = 2$.

Theorem 4.2.3 is an immediate corollary of theorems 1.2 and 1.4 in [1].

Let the function $\iota : F \to F^n$ be defined as $\iota : t \mapsto (t, o)$. With these notion we get that $Tr_m^k = \iota \circ f_m^k$.

The following proposition says that, according any inertial observers, the life-curve of every observer is differentiable and the derivate of it is the time-unit vector of the inertial co-moving observe.

Proposition 4.2.4. Assume AccRel_0 and $m \in IOb, k \in Ob$. Then Tr_m^k is differentiable and $Tr_m^{k\,\prime}(t) = 1_m^{k_t}$. Thus $\mu\big(Tr_m^{k\,\prime}(t)\big) = \pm 1$. Moreover,if k STL m, then $\mu\big(Tr_m^{k\,\prime}(t)\big) = 1$.

proof. Let $m \in IOb$, $k \in Ob$ and $t \in F$; and let k and $k_t \in IOb$ be co-moving observers in $\iota(t) = (t, o)$. Then $Tr_{k_t}^{k\,\prime}(t) = (\iota \circ f_{k_t}^k)'(t) = d_{\iota(t)} f_{k_t}^k\big(\iota'(t)\big) = 1_t$ by Theorem A.0.6 since $\iota'(t) = 1_t$ and $d_{\iota(t)} f_{k_t}^k = Id$ by Proposition 4.2.2. Thus $Tr_m^{k\,\prime}(t) = (Tr_{k_t}^k \circ f_m^{k_t})'(t) = f_m^{k_t}\big(Tr_{k_t}^{k\,\prime}(t)\big) - f_m^{k_t}(o) = f_m^{k_t}(1_t) - f_m^{k_t}(o) = 1_t^{k_t}$ by Corollary A.0.8 since $f_m^{k_t}$ is affine and $Tr_{k_t}^{k\,\prime}(t) = 1_t$. By Theorem 4.2.3, $f_m^{k_t}$ preserves the absolute value of the Minkowski-length, i.e. for all $p, q \in F^n$ $\big|\mu(f_m^{k_t}(p), f_m^{k_t}(q))\big| = |\mu(p,q)|$. Thus $\big|\mu\big(Tr_m^{k\,\prime}(t)\big)\big| = \big|\mu\big(f_m^{k_t}(1_t), f_m^{k_t}(o)\big)\big| = |\mu(1_t, o)| = 1$. If k STL m then $\mu\big(Tr_m^{k\,\prime}(t)\big) \geq 0$. Thus $\mu\big(Tr_m^{k\,\prime}(t)\big) = 1$. This is what we wanted to prove. ∎

Proposition 4.2.5. Assume AccRel. Let $m \in IOb$, $k \in Ob$. Then(i) and (ii) below hold:

(i) Tr_m^k is time-like iff Tr_m^k is STL.

(ii) Tr_m^k is STL or FTL.

proof. (i) If Tr_m^k is time-like, then it is **STL**, by Proposition A.0.15. Tr_m^k is differentiable and $\mu\big(Tr_m^k(t)\big) = 1$ for all $t \in \bar{t}$ by Proposition 4.2.4. Thus, if Tr_m^k is STL, then it is time-like, too.

(ii) By Proposition A.0.15, it is enough to prove that Tr_m^k does not have light-like chords.

In the case when $n = 2$, if Tr_m^k has a light-like chord then it has a light-like derivate by Corollary A.0.13 since a 1-dimensional subspace of F^2 that contains a light-like chord must be a self-orthogonal straight-line. This contradict the fact that Tr_m^k cannot have light-like derivatives by Proposition 4.2.4.

In the case when $n > 2$, Tr_m^k cannot have light-like or space-like derivatives by Proposition A.0.15 and Theorem 2.4.2. Thus it cannot have light-like or space-like chords, either, by Corollary A.0.13 since for every light-like chord $\{p, q\}$ there is an $(n-1)$-dimensional subspace H such that $p - q \in H$. ■

For any $\gamma : F \supset [a, b] \longrightarrow F^n$ STL curve, let the **Minkowski-length** of γ be defined as:

$$\mu(\gamma) := \inf\{\sum_{i=0}^{k} \mu\big(\gamma(q_i) - \gamma(q_{i-1})\big) : a = q_0 < q_1 < \ldots < q_k = b\}.$$

In the definition of the Minkowski-length, we use infimum since in Minkowski geometry the triangle inequality holds in the reverse way in triangles whose sides are STL, cf. the inertial twin paradox.

We call a curve **Minkowski parametrized** if it is parametrized by its Minkowski-length.

The following theorem states that each STL observer's life-curve must be Minkowski parametrized in the models of AccRel if F is the ordered field of the real numbers.

Theorem 4.2.6. If we assume $F = \mathbb{R}$, then

$$\mathsf{AccRel} \models \forall m \in IOb\ k \in Ob\ \ k\ \mathsf{STL}\ m \Longrightarrow \mu(Tr_m^k\big|_{[a,b]}) = b - a.$$

proof. The proof of the theorem is based on the following lemma:

Lemma 4.2.7. Assume AccRel_0. If $m \in IOb$, $k \in Ob$ and k STL m, then for all $\varepsilon \in F^+$ the following holds:

$$\forall q \in F\ \exists \delta \in F^+\ \ \forall p \in (q, q+\delta)\ \ \Big|\mu\big(Tr_m^k(p) - Tr_m^k(q)\big) - (p - q)\Big| \le \varepsilon(p - q).$$

proof. If $p \in F$, then we use the notation $\tilde{p}$ for (p, o) throughout this proof. Notice that this notation implies $\big|\tilde{p} - \tilde{q}\big| = p - q$ if $q < p$. It is enough to prove the lemma in the case when m is the inertial co-moving observer of k in the point $\tilde{q}$, since the Minkowski-distance between two events is the same for all the inertial observers, cf. Theorem 4.2.3.

In this case, we know that

$$Tr_m^k(p) := f_m^k(\tilde{p})\ \text{ and }\ Tr_m^k(q) := f_m^k(\tilde{q}) = \tilde{q}. \tag{*}$$

Since m and k are co-moving at q, we get:

$$\big|Tr_m^k(p) - \tilde{p}\big| = \big|f_m^k(\tilde{p}) - \tilde{p}\big| \le \varepsilon\big|\tilde{p} - \tilde{q}\big|. \tag{**}$$

From these we get:

$$\Big|\big|Tr_m^k(p) - Tr_m^k(q)\big| - \big|\tilde{p} - \tilde{q}\big|\Big| \le \Big|Tr_m^k(p) - Tr_m^k(q) - (\tilde{p} - \tilde{q})\Big| \stackrel{*}{=}$$
$$\Big|Tr_m^k(p) - \tilde{p}\Big| \stackrel{**}{\le} \varepsilon\big|\tilde{p} - \tilde{q}\big| = \varepsilon(p - q) \tag{4.1}$$

if $|p - q| < \delta$ for some δ since $\big||a| - |b|\big| \le |a - b|$ holds for all $a, b \in F^n$.

The following inequality holds

$$\Big|\mu\big(Tr_m^k(p) - Tr_m^k(q)\big) - \big|Tr_m^k(p) - Tr_m^k(q)\big|\Big| \le \varepsilon\,\Big|Tr_m^k(p) - Tr_m^k(q)\Big| \tag{4.2}$$

if $|p-q| < \delta$ for some δ since the quotient of the Minkowski- and the Euclidean-length of some vector tends to 1 as the vector tends to vertical and $Tr^k_m(p) - Tr^k_m(q)$ tends to vertical as p tends to q.

From the triangle inequality, we get:

$$\left|Tr^k_m(p) - Tr^k_m(q)\right| \overset{*}{\leq} \left|Tr^k_m(p) - \tilde{p}\right| + \left|\tilde{p} - \tilde{q}\right| \overset{**}{\leq} (\varepsilon + 1)\left|\tilde{p} - \tilde{q}\right| = (\varepsilon + 1)(p - q). \tag{4.3}$$

From inequalities 4.2 and 4.3, we get:

$$\left|\mu\big(Tr^k_m(p) - Tr^k_m(q)\big) - \left|Tr^k_m(p) - Tr^k_m(q)\right|\right| \leq \varepsilon(\varepsilon + 1)(p - q). \tag{4.4}$$

From inequalities 4.1 and 4.4, we can easily derive the desired one by using the triangle inequality. Thus we completed the proof of the lemma. ■

We return to the proof of Theorem 4.2.6. Let $\mathcal{Q} = \{q_i : a = q_0 < q_1 < \ldots < q_k = b\}$ be an arbitrary partition of the interval $[a,b]$. For every $p \in [a,b]$ there is some δ_p such that Lemma 4.2.7 holds for ε. Since F is the ordered field of the reals, $[a,b]$ is compact. So we can choose finite many of these p's such as $B(p) := B_{\delta_p}(p)$'s cover $[a,b]$. Let us choose partition $\mathcal{P} = \{p_i : a = p_0 < p_1 \ldots < p_{2n} = b\}$ such that $B(p_i)$'s cover $[a,b]$, all q_i's are among the p_i's and $p_{2i+1} \in [a,b] \cap B(p_{2i}) \cap B(p_{2i+2})$. Thus we get a refinement of partition $\mathcal{Q}$ such that $\left|\mu\big(Tr^k_m(p_{i+1}) - Tr^k_m(p_i)\big) - (p_{i+1} - p_i)\right| \leq \varepsilon(p_{i+1} - p_i)$ hold for all $p_i, p_{i+1} \in \mathcal{P}$.

By adding these inequalities and applying the triangle inequality, we get the following:

$$\left|\sum_{i=0}^{2n} \mu\big(Tr^k_m(p_{i+1}) - Tr^k_m(p_i)\big) - (b - a)\right| \leq$$

$$\sum_{i=0}^{2n} \left|\mu\big(Tr^k_m(p_{i+1}) - Tr^k_m(p_i)\big) - (p_{i+1} - p_i)\right| \leq \sum_{i=0}^{2n} \varepsilon(p_{i+1} - p_i) = \varepsilon(b - a).$$

Since ε was an arbitrary element of F^+ and $\mathcal{Q}$ was an arbitrary partition of $[a,b]$, we get that $\mu(Tm^k_m\big|_{[a,b]}) = b - a$ as we wanted. ■

Question 2. Does Theorem 4.2.6 remain valid if F is not the ordered field of the reals? We conjecture that the answer is yes; but it is not straightforward to prove it, since the Minkowski-length of a curve is not a first-order definition therefore IND cannot be used here directly.

4.3 Twin paradox with accelerated observers

The function $\pi_t : F^n \to F$ is defined as $\pi_t : p \mapsto p_t$. Let the function $\tau^k_m : F \to F$ be defined as

$$\tau^k_m := \iota \circ f^k_m \circ \pi_t = Tr^k_m \circ \pi_t.$$

The following theorem is the key theorem for proving that the twin paradox is true in the models of Accrel, cf. Theorem 4.3.2.

Theorem 4.3.1. Assume AccRel, $x, y \in F$, $x < y$ and $m \in IOb$, $k \in Ob$ are such observers that k STL m. Then $y - x \leq \left|\tau^k_m(y) - \tau^k_m(x)\right|$. Moreover, if $Tr^k_m(p)_s \neq Tr^k_m(q)_s$ for some $p, q \in [x,y]$, then $y - x < \left|\tau^k_m(y) - \tau^k_m(x)\right|$.

proof. Throughout this proof, we omit the sub and superscripts of τ_m^k and denote Tr_m^k with γ. $\tau'(t) := (\gamma \circ \pi_t)'(t) = \pi_t(\gamma'(t))$ by Corollary A.0.8 since π_t is a linear map. $\mu(\gamma'(t)) = 1$ by Proposition 4.2.4. Thus $|\tau'(t)| = |\pi_t(\gamma'(t))| \geq 1$ and equal with 1 iff $\gamma'(t)$ is vertical since a the first coordinate of a Minkowski-one-length vector is always greater-than-one and equals one iff the named vector is vertical. Thus if there were some $x, y \in F$ such that $x < y$ and $y - x > |\tau(y) - \tau(x)|$, then from Theorem A.0.11 there were some $t \in (x, y)$ such that $|\tau'(t)| = |\frac{\tau(y)-\tau(x)}{y-x}| < 1$. This contradicts $|\tau'(t)| \geq 1$. Thus $y - x \leq |\tau(y) - \tau(x)|$. So we have completed proving the first statement of the theorem.

It is clear that τ is injective since γ is STL. From Theorem A.0.3 it is clear that τ is monotonous, i.e. $[\forall x, y \in F \;\; x < y \Rightarrow \tau(x) < \tau(y)] \vee [\forall x, y \in F \;\; x < y \Rightarrow \tau(y) < \tau(x)]$, since it is continuous and injective. For proving the second part of the theorem, let us assume that there are some $p, q \in [x, y]$ where $\gamma(p)_s \neq \gamma(q)_s$ (we can assume that $p < q$). Thus form Corollary A.0.13, we get that there is some $t \in (p, q)$ such that $\gamma'(t)$ is not vertical. Thus $|\tau'(t)| > 1$ and therefore there must be some $r, s \in (p, q)$ such that $r > s$ and $r - s < |\tau(r) - \tau(s)|$. Since τ is monotonous, we get that $y - x < |\tau(y) - \tau(x)|$ by adding the inequalities $y - r \leq |\tau(y) - \tau(r)|$, $r - s < |\tau(r) - \tau(s)|$ and $s - x \leq |\tau(s) - \tau(x)|$. This completes the proof of the second statement. ■

Let us formulate the accelerated version of the twin paradox. We say that two observers a, i are in **twin paradox relation** at coordinate points p, q in the coordinate domain of observer m if:

$$\mathsf{Twp}_m(a \leq i)(p, q) :\Longleftrightarrow \mathsf{Time}_a^m(p, q) \leq \mathsf{Time}_i^m(p, q).$$

$\mathsf{AxTwpAcc}$ The inertial observers measure more or equal time between two meeting points than the other observers:

$$\forall m, i \in IOb \;\; \forall a \in Ob \;\; \forall p, q \in \mathsf{CD}(m) \;\; \mathsf{meet}_m^q(a, i) \wedge \mathsf{meet}_m^p(a, i) \Longrightarrow \mathsf{Twp}_m(a \leq i)(p, q).$$

Theorem 4.3.2. $\mathsf{AccRel} \models \mathsf{AxTwpAcc}$.

proof. From the definition of the relation Time, it follows that we can assume that $m = i$, cf. the remark on page 13. In the case when $m = i$, Tr_i^a is STL by Proposition 4.2.5 since it has a vertical chord. Therefore a STL i. Thus we can use Theorem 4.3.1.

We use it in the following way: Let $x := f_a^i(q)_t$ and $y := f_a^i(p)_t$. We can assume that $x < y$. It is easy to see that $p_t = \pi_t(p) = \tau_i^a(y)$ and $q_t = \pi_t(q) = \tau_i^a(x)$ since $p, q \in \bar{t}$. Thus

$$\mathsf{Time}_a^i(p, q) = |f_a^i(p)_t - f_a^i(q)_t| = y - x \leq |\tau_i^a(y) - \tau_i^a(x)| = |p_t - q_t| = \mathsf{Time}_i^i(p, q).$$

This completes the proof of the theorem. ■

If we assume that F is the ordered field of the reals, then Theorem 4.3.2 is an immediate corollary of Theorem 4.2.6 since between two points the straight-line has the longest Minkowski-length among all the STL lines.

Without IND, Theorem 4.3.2 does not remain true if F is not the ordered field of the reals. Since there are "holes" in the other ordered fields and in such a "hole" the observers clock can jump ahead and ruin the twin paradox. The axiom scheme IND is a tool for filling this kind of "holes" but it can fill only the first-order definable ones.

4.4 Two possible ways for constructing accelerated world-views

In this section, we introduce two observer-adding construction steps and formulate two conjectures about them.

We call **radar construction** the following one: Let $\mathfrak{M}$ be a two-dimensional model of $\mathsf{Specrel}_0$ or $\mathsf{Specrel}^+$, let m be an observer in $\mathfrak{M}$ and let $\gamma : F \longrightarrow \mathsf{CD}(m)$ be a curve that intersects every self-orthogonal straight-line at most once. We construct a new model $\mathfrak{M}^{\mathsf{rad}}_{m,\gamma}$ where γ is going to be the life-curve of the only new observer k in m's coordinate domain. Let us choose the f^k_m transformation such that it takes parallel self-orthogonal straight-lines to parallel self-orthogonal straight-lines and let $(p, q) \in f^k_m$ iff p is the intersection of two self-orthogonal straight-lines through $(\tau - t, 0)$ and $(\tau + t, 0)$ while q is the intersection of two self-orthogonal straight-lines through $\gamma(\tau - t)$ and $\gamma(\tau + t)$, cf. Figure 4.1. Let the event function of m in $\mathfrak{M}^{\mathsf{rad}}_{m,\gamma}$ be $ev^+_m(p) := ev_m(p)$ if $p \notin Rng(\gamma)$ and $ev^+_m(p) := ev_m(p) \cup \{k\}$ if $p \in Rng(\gamma)$. Let $ev^+_h := f^h_m \circ ev^+_m$ be the event function of an other observer h.

It is easy to see that the radar construction preserves the axioms AxEvTr, AxPhTr and AxSelf.

Question 3. For what kind of curves, can the radar construction be extended in higher dimensions?

We call a transformation of F^n **future preserving Poincaré transformation** if it is a Poincaré transformation which takes future light-cones to future light-cones. We use the symbol Poi^+ for the set of the future preserving Poincaré transformations.

AxPoi^+ For every inertial observer m and future preserving Poincaré transformation f, there is an inertial observer k such that the world-view transformation between m and k is f

$$\forall m \in IOb \ \forall f \in \mathsf{Poi}^+ \ \exists k \in IOb \ \ f^k_m = f.$$

Conjecture 4. If we assume $F = \mathbb{R}$, $\mathfrak{M} \models \mathsf{AccRel} \cup \{\mathsf{AxPoi}^+\}$, $m \in IOb_2$ and $\gamma : F \longrightarrow \mathsf{CD}(m)$ is a well-parametrized time-like curve, then $\mathfrak{M}^{\mathsf{rad}}_{m,\gamma} \models \mathsf{AccRel} \cup \{\mathsf{AxPoi}^+\}$.

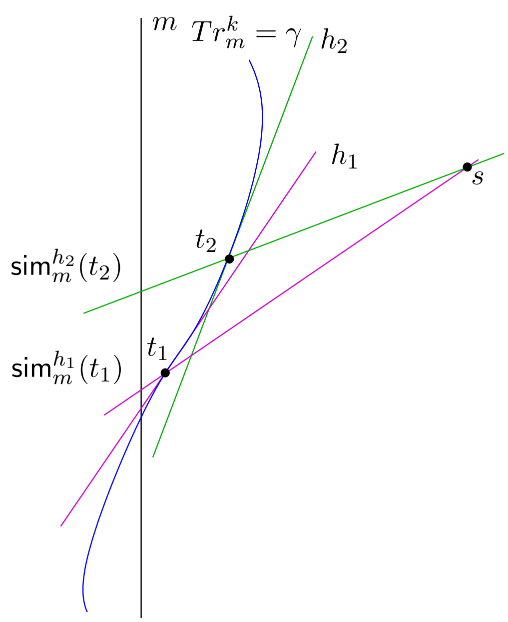

Figure 4.5: for the tangent construction.

There is an another natural way of building a new observer's world-view from a time-like curve in some inertial observers word view. The following construction is called **tangent construction**: Let $\mathfrak{M}$ be a model of $\mathsf{Specrel}^+$, let m be an inertial observer and let $\gamma : F \longrightarrow \mathsf{CD}(m)$ be a time-like curve. Like in the radar construction, we construct a new model $\mathfrak{M}^{\mathsf{tan}}_{m,\gamma}$ where γ is going to be the life-curve of the only new observer k in m's coordinate domain. Let the f_m^k transformation be constructed by the following way: Let $Tr_m^k := \gamma$. Since γ is differentiable and since AxPoi^+ there is some inertial observer, say h, whose life-line tangents γ and well-configured to k at the tangent point. (The well-configuredness has meaning in this situation since $ev_k(0)$ has already been defined.) Let the simultaneity of k through this tangent point of Tr_m^k be the same as the simultaneity of h through this point, cf. Figure 4.5. Let the event function of m in $\mathfrak{M}_{m,\gamma}^{\mathsf{tan}}$ be $ev_m^+(p) := ev_m(p)$ if $p \notin Rng(\gamma)$ and $ev_m^+(p) := ev_m(p) \cup \{k\}$ if $p \in Rng(\gamma)$. Let $ev_h^+ := f_m^h \circ ev_m^+$ be the event function of an other observer h.

Notice that if the trace of the observer is not a straight-line than his simultaneities have an intersection, i.e. the world-view transformation is not injective, cf. Figure 4.5.

Conjecture 5. If we assume $F = \mathbb{R}$, $\mathfrak{M} \models \mathsf{AccRel} \cup \{\mathsf{AxPoi}^+\}$, $m \in IOb$ and $\gamma : F \longrightarrow \mathsf{CD}(m)$ is a continuously differentiable well-parametrized time-like curve, then $\mathfrak{M}_{m,\gamma}^{\mathsf{tan}} \models \mathsf{AccRel} \cup \{\mathsf{AxPoi}^+\}$.

4.5 The uniformly accelerated observer

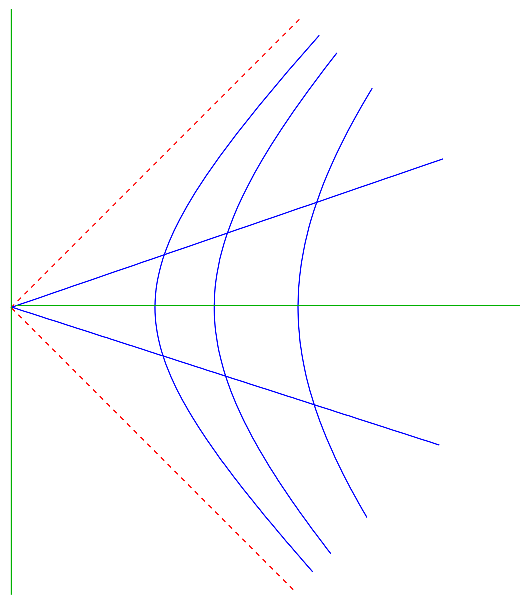

Figure 4.6: the coordinate system of the uniformly accelerated observer.

Appendix A

Analysis over arbitrary ordered fields

In this chapter, we recall some statements and definitions from real analysis and generalize them for arbitrary ordered fields by using the axiom scheme IND. The power of this axiom scheme can easily be illustrated by the next proposition.

Let $\mathfrak{M}$ be a frame model. An n-ary relation $R \subseteq F^n$ is said to be **definable** in $\mathfrak{M}$ iff there is a formula φ with only free variables $x_1, \ldots, x_n, y_1, \ldots, y_k$ and there are $a_1, \ldots, a_k$ in the universe of $\mathfrak{M}$ such that $R = \{(p_1, \ldots, p_n) \in F^n : \mathfrak{M} \models \varphi(p_1, \ldots, p_n, a_1, \ldots, a_k)\}$. Notice that IND says that every non empty, bounded and definable subset of F has a supremum.

Proposition A.0.1. $\mathsf{IND} \models F$ is real-closed.[1]

proof. Let $p(y)$ be the odd degree polynomial $a_{2n+1}y^{2n+1} + \ldots + a_1 y + a_0$. It is enough to prove that $p(y)$ has a root when $a_{2n+1} > 0$. Let $H := \{t \in F : p(t) < 0\}$. It is clear that H is not empty, bounded and definable. From IND, it follows that H has a supremum, say s. Both $\{t : p(t) > 0\}$ and $\{t : p(t) < 0\}$ are open sets, since $p(y)$ is continuous. Thus $p(s)$ cannot be negative since s is an upper bound of H, and cannot be positive since s is the smallest upper bound, i.e. $p(s) = 0$ as desired.

Let a be a positive element of F and let $H := \{y \in F : y^2 < a\}$. Then H is not empty, bounded and definable. From IND, it follows that H has a supremum and from the same reasons as before this supremum is a square root of a. ■

Let $a, b, c \in F$. We say that b is **between** a and c iff $a < b < c$ or $a > b > c$. In this case, we write $\mathsf{Bw}(a, b, c)$. We use the following notations: $[a, b] := \{t \in F : a \leq t \leq b\}$, $(a, b) := \{t \in F : a < t < b\}$, $[a, b) := \{t \in F : a \leq t < b\}$ and $(a, b] := \{t \in F : a < t \leq b\}$.

Convention 5. Whenever we write $[a, b]$, we assume that $a, b \in F$ and $a < b$. We also use this convention for $[a, b)$, $(a, b]$ and (a, b).

Lemma A.0.2 (Cousin's lemma). Assume IND. Let $\mathcal{A}$ be a set of some subintervals of $[a, b]$ which has the following properties:

(i) **beginable**: for each $x \in [a, b]$, $\mathcal{A}$ contains every small enough right and left neighborhood of x, i.e. $\forall x \in [a, b]\ \exists c, d \in F\ \ c < x < d\ \ \forall y \in [c, d] \cap [a, b]\ \ (y < x \Rightarrow [y, x] \in \mathcal{A}) \wedge (x < y \Rightarrow [x, y] \in \mathcal{A})$.

(ii) **connectable**: if $[x, y], [y, z] \in \mathcal{A}$ then $[x, z] \in \mathcal{A}$,

[0] Throughout this section, we used only that $\langle F, \leq, \cdot, /, +, - \rangle$ is an (linearly) ordered (commutative) field and the symbols $\leq, \cdot, /, +, -$ are in the language of $\mathsf{IND_L}$.

[1] An ordered field F is called **real-closed** if every positive element has a square root and every polynomial of odd degree has a root.

(iii) **definable**: the set $\{t \in F : [a,t] \in \mathcal{A}\}$ is definable.

Then $[a,b] \in \mathcal{A}$.

proof. From IND, it follows that the set $H := \{x \in F : a < x \wedge \forall t \in (a,x) \quad [a,t] \in \mathcal{A}\}$ has a supremum since it is a definable, non-empty (since $\mathcal{A}$ is beginable) and bounded set. Let us call this supremum s. We complete the proof by proving that $[a,s] \in \mathcal{A}$ and $s = b$.

Since $\mathcal{A}$ is beginable, there is a $c \in [a,s)$ such that $[c,s] \in \mathcal{A}$. Since s is the supremum of H, for all $t \in (a,s) \quad [a,t] \in \mathcal{A}$. Thus $[a,c] \in \mathcal{A}$. Thus by the connectability of $\mathcal{A}$, we get that $[a,s] \in \mathcal{A}$.

If $s < b$, then there is an $e \in (s,b]$ such that $[s,t] \in \mathcal{A}$ for all $t \in (s,e]$ since $\mathcal{A}$ is beginable. Thus we get that for all $t \in (s,e] \quad [a,t] \in \mathcal{A}$ by using the connectability of $\mathcal{A}$ and the fact that $[a,s] \in \mathcal{A}$. But then, for all $t \in (a,e] \quad [a,t] \in \mathcal{A}$. This contradicts the fact that s is the supremum of the set H therefore $s = b$.[2] ■

A set $H \subseteq F$ is called **open** if $\forall x \in H \ \exists a,b \in H \quad x \in (a,b) \subset H$. The open sets of F form a topology. This topology is called the **order-topology**. A function $f : [a,b] \to F$ is called **continuous** if the inverse image of every open subinterval of F is open, i.e. $\{x : f(x) \in (c,d)\}$ is open for all $c,d \in F$.

Theorem A.0.3 (Boltzano's Theorem). Assume IND. Let $f : [a,b] \to F$ be a definable continuous function. If $f(a) < c < f(b)$, then there is a $t \in [a,b]$ such that $f(t) = c$.

proof. Let $\mathcal{A} := \{[x,y] \subseteq [a,b] : (\forall t \in [x,y] \quad f(t) < c) \vee (\forall t \in [x,y] \quad f(t) > c)\}$ and assume that there is no such $t \in [a,b]$ that $f(t) = c$. $\mathcal{A}$ is definable since f is definable. $\mathcal{A}$ is beginable since f is continuous. The connectability of $\mathcal{A}$ is also clear. Thus from Cousin's lemma we get that $\forall t \in [a,b] \quad f(t) < c$ or $\forall t \in [a,b] \quad f(t) > c$. So if $f(a) < c$ and $f(b) > c$, then there must be some t where $f(t) = c$. This completes the proof of the theorem.[2] ■

Theorem A.0.4. Assume IND. Let $f : [a,b] \to F$ be a definable continuous function. Then $\sup\{f(x) : x \in [a,b]\}$ exists and there is a $t \in [a,b]$ where $f(t) = \sup\{f(x) : x \in [a,b]\}$.

proof. Let $H := \{f(x) : x \in [a,b]\}$ and $\mathcal{A} := \{[x,y] \subseteq [a,b] : \exists c \in F \ \ \forall t \in [x,y] \quad f(t) < c\}$. Since $\mathcal{A}$ is definable, beginable and connectable therefore H is bounded by Cosine's Lemma. Thus from IND it follows that $\sup H$ exists since H is nonempty, definable and bounded. If there is no $t \in [a,b]$ such that $f(t) = \sup H$, then $\mathcal{A} := \{[x,y] \subseteq [a,b] : \exists q \in F \ \ \forall t \in [x,y] \quad f(t) < q < \sup H\}$ is also definable, beginable and connectable. Thus $[a,b] \in \mathcal{A}$ by Cousin's lemma. Thus there is a $q < \sup H$ such that $f(t) < q$ for all $t \in [a,b]$ and this contradicts the supremum property. This completes the proof of the theorem.[2] ■

A function f from $H \subseteq F^n$ to F^k is called **continuous** at $q \in H$ if the usual formula of continuity holds for f, i.e.:

$$\forall \varepsilon \in F^+ \ \exists \delta \in F^+ \ \ \forall p \in H \quad |q - p| < \delta \Longrightarrow |f(q) - f(p)| < \varepsilon.$$

The function f is called continuous if it is continuous at every $q \in H$; and f is called **uniformly continuous** if it is continuous and the same δ is good for every $q \in H$ for a given ε.

We call a set $G \subseteq F^n$ **open** iff for all $p \in G$ there is an $\varepsilon \in F^+$ such that $B_\varepsilon(p) \subset G$. Let $H \subseteq F^n$. The **interior** of H is defined as $int(H) := \{p \in H : \exists \varepsilon \in F^+ \ \ B_\varepsilon(p) \subset H\}$. We call a set $Z \subseteq F^n$ **closed** iff $F^n \setminus Z$ is open. Notice that $\{p\}$ is open for all $p \in F^n$.

[2]In the proof of Lemma A.0.2 and Theorems A.0.3 and A.0.4, we used only that $\langle F, \leq \rangle$ is an (linearly) ordered set and the language of IND_L contains the binary relation symbol $\leq$ and the unary relation symbol F.

Proposition A.0.5. Let $f : F^n \longrightarrow F^m$. The following three statement are equivalent:

(i) f is continuous.

(ii) The f^{-1}-image of a closed set is closed.

(iii) The f^{-1}-image of a open set is open. ■

It is well known and also easy to see that this definition and the definition mentioned earlier for the continuity is the same for functions from $[a, b] \subset F$ to F.

We say that function $f : [a, b] \longrightarrow F$ is **locally maximal** at $x \in (a, b)$ iff there is a $\delta \in F^+$ such that $f(y) \leq f(x)$ for all $y \in (x - \delta, x + \delta)$. The **local minimality** is defined analogously.

We can generalize the definitions of differentiability and the limit of a function in a similar way to the continuity. We say that function f from $H \subseteq F^n$ to F^k **tends to** $a \in F^k$ while $p \in H$ tends to $q \in H$ if the usual formula for the limit of a function holds for f:

$$\forall \varepsilon \in F^+ \ \exists \delta \in F^+ \ \ \forall p \in H \ \ 0 < |q - p| < \delta \Longrightarrow |a - f(p)| < \varepsilon.$$

This a is unique iff q is not an isolated point of H, i.e. $\forall \varepsilon \in F^+ \ \ B^\circ_\varepsilon(q) \cap H \neq \emptyset$. In this case, we call a the **limit** of the function f in the point q and we write that $\lim_{p \to q} f(p) = a$.

Let $H \subseteq F^n$. We say that a function $f : H \longrightarrow F^m$ is **differentiable** in $q \in H$ if the usual formula

$$\forall \varepsilon \in F^+ \ \exists \delta \in F^+ \ \ \forall p \in H \cap B_\delta(q) \ \ \ |f(p) - f(q) - A(p - q)| \leq \varepsilon \, |p - q|$$

holds for some linear map $A : F^n \longrightarrow F^m$. This a A is unique if, q is in the interior of H, i.e. $q \in int(H)$. In this situation, call A the **differential** of f and denote it by $d_q f$.

In the case when f is a function from $H \subseteq F$ to F^k, the differential of f, if it exists and is unique, can be defined as the limit of the function $h \mapsto \frac{f(q+h)-f(q)}{h}$, i.e. as $\lim_{h \to 0} \frac{f(q+h)-f(q)}{h}$. In this situation, the differential of f in $q \in H$ is not a linear map but a vector. In this case, we use the notation $f'(q)$ for the differential and we call it the **derivate vector** of f in q. The connection between the two definition is the following: $d_q f(t) = t f'(q)$ where $t \in F$.

Notice that the basic properties of the limit and the differential are true over any ordered field since we used only the ordered field property of the real numbers while we were proving them.

The proof of the following theorem also uses only the ordered field property of the real numbers, cf., e.g. [18].

Theorem A.0.6 (chain rule). Let $g : F^n \to F^m$ and $f : F^m \to F^k$. If g is differentiable in $t \in F^n$ and f is differentiable in $g(t)$, then $g \circ f$ is differentiable in t and its differential is $d_t g \circ d_{g(t)} f$, i.e.

$$d_t(g \circ f) = d_t g \circ d_{g(t)} f.$$

in particular if $g : F \to F^m$ and $f : F^m \to F^k$, then

$$(g \circ f)'(t) = d_{g(t)} f\big(g'(t)\big).$$

proof. The proof is the same as in real analysis, cf., e.g. [18, Theorem 5.5]. ■

Proposition A.0.7. The differential of an affine map is its linear part; i.e. if $A : F^n \to F^m$ is an affine map, then $d_q A(p) = A(p) - A(o)$, where $q, p \in F^n$ and o is the origin of F^n.

proof. The proof is straightforward from the definitions. ■

Corollary A.0.8. If $g : F \to F^n$ and $A : F^n \to F^m$ is an affine map, then $(g \circ A)'(t) = A\big(g'(t)\big) - A(o)$. ■

Proposition A.0.9. If a function $f : [a,b] \to F$ is locally maximal (minimal) and differentiable in $x \in (a,b)$, then its differential is 0 at x, i.e. $f'(x) = 0$.

proof. The proof is the same as in real analysis, cf., e.g. [18, Theorem 5.8]. ■

Theorem A.0.10 (Darboux's Theorem). Assume IND. Let $f : F \to F$ be a definable and differentiable function and let $a, b \in F$. Then for all $d \in F$ between $f'(a)$ and $f'(b)$ there is a $c \in F$ between a and b such that $f'(c) = d$.

proof. We can assume that $f'(a) > d > f'(b)$. Let $g(t) := f(t) - td$. Then g is differentiable and $g'(a) > 0$, $g'(b) < 0$. Thus g cannot be maximal in a or b by Proposition A.0.9. Thus, from Theorem A.0.4, we get that there is a point, say c, between a and b where g is maximal. Thus also form Proposition A.0.9, we get that $g'(c) = f'(c) - d = 0$. ■

Theorem A.0.11 (Mean Value Theorem). Assume IND. Let $f : [a,b] \to F$ be a definable continuous function which is differentiable on (a,b). Then there is at least one point $t \in (a,b)$ such that $f'(t) = \frac{f(b)-f(a)}{b-a}$.

proof. Let $h(t) := \big(f(b) - f(a)\big)t - (b-a)f(t)$. Then h is continuous on $[a,b]$, differentiable on (a,b) and $h(a) = f(b)a - bf(a) = h(b)$. If h is constant then $h'(t) = 0$ for all $t \in (a,b)$. Otherwise there is a maximum/minimum of h different from $h(a) = h(b)$ in some $t \in (a,b)$. Hence $h'(t) = 0$ by Theorem A.0.9. This completes the proof since $h'(t) = f(b) - f(a) - (b-a)f'(t)$. ■

Corollary A.0.12 (Rolle's Theorem). Assume IND. Let $f : [a,b] \to F$ be a definable continuous function which is differentiable on (a,b). If $f(a) = f(b)$, then there is at least one point $t \in (a,b)$ such that $f'(t) = 0$. ■

Corollary A.0.13. Assume IND. Let $\gamma : F \longrightarrow F^n$ be a definable and differentiable curve. Then for all distinct $a, b \in F$ and for every $(n-1)$-dimensional subspace H that contains $\gamma(a) - \gamma(b)$, there is at least one c between a and b such that $\gamma'(c)$ is in H.

proof. The derivate vector of a curve γ composed by a linear map A in $t \in F$ is the A-image of $\gamma'(t)$ by Corollary A.0.8. Since any $(n-1)$-dimensional subspace of F^n can be taken to $\{0\} \times F^{n-1}$ by a linear transformation, we can assume that $H = \{0\} \times F^{n-1}$. Recall that the function $\pi_t : F^n \longrightarrow F$ is defined as $p \mapsto p_t$. Then $\gamma \circ \pi_t(a) = \gamma \circ \pi_t(b)$ since $\gamma(a) - \gamma(b) \in H$. By applying the Rolle's Theorem to $\gamma \circ \pi_t$, we get that there is a $c \in F$ such that $(\gamma \circ \pi_t)'(c) = 0$. Thus $\gamma'(c)$ is in with H since $(\gamma \circ \pi_t)'(c) = \pi_t\big(\gamma'(c)\big) = \gamma'(c)_t$ by Corollary A.0.8. ■

Proposition A.0.14. Assume IND. Let $f : (a,b) \to F$ be a definable differentiable function for some $a, b \in F$. If $f'(t) = 0$ for all $t \in (a,b)$ then there is a $c \in F$ such that $f(t) = c$ for all $t \in (a,b)$.

proof. If there are $x, y \in (a,b)$ such that $f(x) \neq f(y)$ and $x \neq y$, then from Theorem A.0.11 there is a t between x and y such that $f'(t)(y-x) = f(y) - f(x) \neq 0$ and this contradicts that $f'(t) = 0$. ■

Proposition A.0.15. Assume IND. Let $\gamma : F \longrightarrow F^n$ be a definable and continuous curve. Then (i) and (ii) below hold:

(i) γ is time-like $\Longrightarrow$ γ is STL.

(ii) γ is STL or FTL or it has a light-like chord.

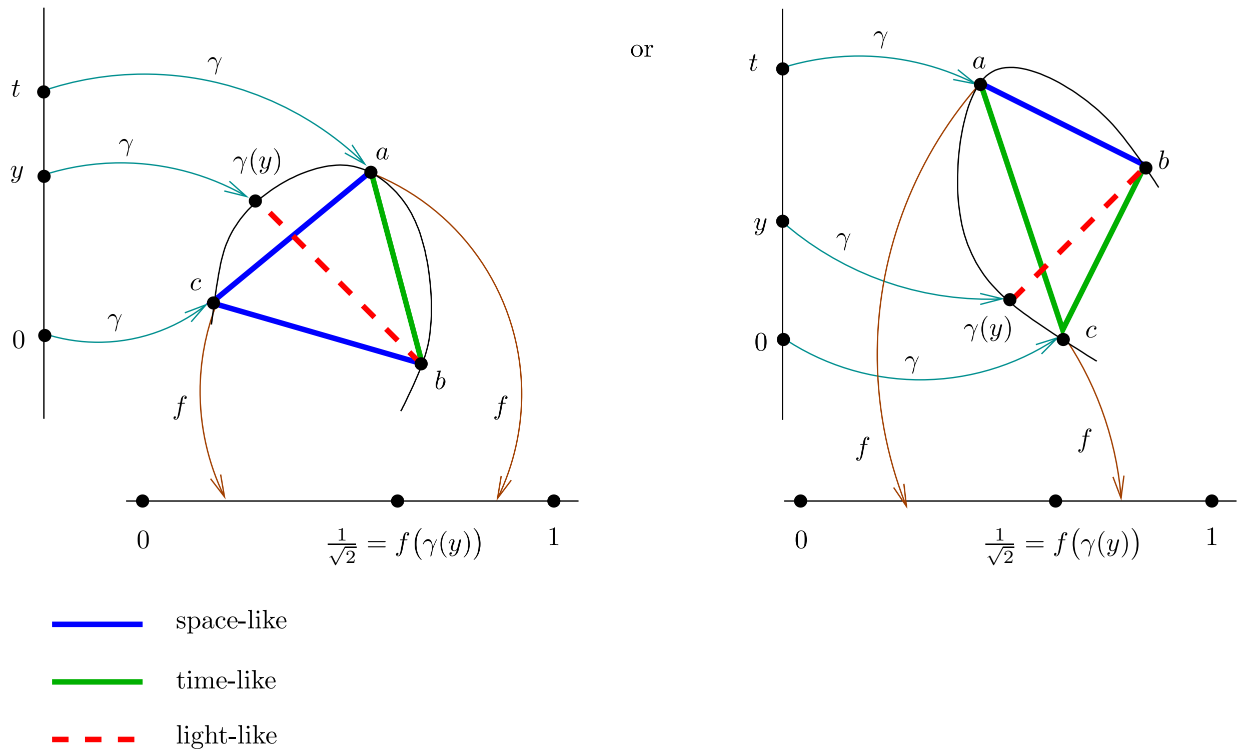

Figure A.1:

proof. To prove the first statement, let us assume that γ is not STL. Then it has a light-like or space-like chord, say $\{p, q\}$. Let H be a $(n-1)$-dimensional subspace that contains $p-q$ and does not contain time-like vectors. Thus, by Corollary A.0.13, we get that there is $t \in F$ such that $\gamma'(t)$ is in H. Since H does not contain time-like vectors, $\gamma'(t)$ is not time like. Thus γ is not time-like.

To prove the second statement, let us assume that γ is not STL or FTL and does not have a light-like chord. Then γ has both time-like and space-like chords. Then there are distinct points $a, b, c \in Rng(\gamma)$ such that the triangle $\{a, b, c\}$ determines two time-like and one space-like or two space-like and one time-like chords of γ. We can assume that $\gamma(0) = c$ and c is the intersection of the chords that have same type. See Figure A.1.

For every $p \in Rng(\gamma)$, by IND, there is a closest $t \in F$ to 0 such that $\gamma(t) = p$, i.e. the set $H := \{\, |x| \,:\, \gamma(x) = p \,\}$ has a minimal element.[3] Thus there is a $t \in F$ such that $\gamma(t)$ is a or b and there is no t' between 0 and t such that $\gamma(t')$ is a or b. We can assume that $\gamma(t) = a$ and $t > 0$.

Let $f : F^n \setminus \{b\} \longrightarrow F$ be the function defined as $p \mapsto \frac{|(p-b)_t|}{|p-b|}$. It is easy to see that f is continuous and for all $p \in F^n \setminus \{b\}$

$$\begin{aligned} f(p) = 1/\sqrt{2} \quad &\Longleftrightarrow \quad p-b \text{ is light-like},\\ f(p) > 1/\sqrt{2} \quad &\Longleftrightarrow \quad p-b \text{ is time-like},\\ f(p) < 1/\sqrt{2} \quad &\Longleftrightarrow \quad p-b \text{ is space-like}. \end{aligned} \tag{A.1}$$

Consider the function $g := \gamma\big|_{[0,t]} \circ f$. It is a continuous function. Furthermore, $Dom(g) = [0, t]$ since there is no $t' \in [0, t]$ such that $\gamma(t') = b$. By (A.1) above and by the fact that $\gamma(0) = c$ and

[3] This is so because of the following. Let s be the supremum of the non-empty bounded definable set $\{\, -|x| \,:\, \gamma(x) = p \,\}$. By the continuity of γ, one of $\gamma(s)$ and $\gamma(-s)$ must be p. But then $-s$ is the minimal element of H.

$\gamma(t) = a$, we have

$$\big(g(0) > 1/\sqrt{2} \text{ and } g(t) < 1/\sqrt{2}\,\big) \;\text{ or }\; \big(g(0) < 1/\sqrt{2} \text{ and } g(t) > 1/\sqrt{2}\,\big)$$

because one of the chords $\{b, c\}$, $\{b, a\}$ is time-like and the other is space-like. But then, by Bolzano's theorem, there is $y \in [0, t]$ such that $g(y) = 1/\sqrt{2}$. For this y, by (A.1) above, we have that $\gamma(y) - b$ is light-like. But then $\{b, \gamma(y)\}$ is a light-like chord of γ. This contradiction proves our proposition. ∎

Index

Bibliography

[1] H. Andréka, J. X. Madarász, and I. Németi. Logical axiomatizations of space-time; samples from the literature. *János Bolyai Conference on Hyperbolic Geometry*, page 32, 2003. http://www.math-inst.hu/pub/algebraic-logic/lstsamples.ps.

[2] H. Andréka, J. X. Madarász, and I. Németi. Logical analysis of relativity theories. *First-Order Logic Revisited, Logos Verlag Berlin*, page 30, 2004. http://www.math-inst.hu/pub/algebraic-logic/foundrel03nov.ps.

[3] H. Andréka, J. X. Madarász, and I. Németi. On the logical structure of relativity theories. Research report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, July 5, 2002. With contributions from A. Andai, G. Sági, I. Sain and Cs. Tőke. http://www.math-inst.hu/pub/algebraic-logic/Contents.html. 1312 pp.

[4] H. Andréka, J. X. Madarász, I. Németi, and G. Székely. A logical investigation of inertial and accelerated observers in flat space-time. In *Kalmár Workshop on Logic and Computer Science*, pages 45–57, 2003.

[5] H. Andréka and I. Németi. Questions about kerr and kerr-newman spacetimes. http://www.math-inst.hu/pub/algebraic-logic/kerr.ps, 2004.

[6] R. d'Inverno. *Introducing Einstein's Relativity*. Oxford University Press, 1983.

[7] J. Earman. *Bangs, Crunches, Whimpers, and Shrieks (Singularities and Acausalities in Relativistic Spacetimes)*. Oxford University Press, New York, Oxford, 1995.

[8] J. Earman and C. Wüthrich. Time machines. Stanford Encyclopedia of Philosophy. http://plato.stanford.edu/contents.html.

[9] G. Etesi and I. Németi. On the effective computability of non-recursive functions via Malament-Hogarth space-times. *International Journal of Theoretical Physics*, pages 341–370, 2002. http://www.bu.edu/wcp/Papers/Scie/ScieDarv.htm.

[10] R. Goldblatt. *Orthogonality and Spacetime Geometry*. Springer-Verlag, 1987.

[11] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, 1973.

[12] M. Hogarth. *Predictability, Computability, and Spacetime*. PhD thesis, University of Cambridge (UK), 2000.

[13] M. Hogarth. Deciding arithmetic in Malament-Hogarth spacetimes, 2001. University of Cambridge (UK), preprint.

[14] J. X. Madarász. *Logic and Relativity (in the light of definability theory)*. PhD thesis, ELTE, Budapest, http://www.math-inst.hu/pub/algebraic-logic/Contents.html., 2002.

[15] J. X. Madarász, I. Németi, and Tőke Cs. On generalizing the logic-approach to space-time towards general relativity: first steps. *First-Order Logic Revisited, Logos Verlag Berlin*, 2004. http://www.math-inst.hu/pub/algebraic-logic/loc-mnt04.ps.

[16] I. D. Novikov. *The river of time*. Cambridge University Press, Cambridge, England, 1998.

[17] B. Pécsi and I. Sain. All-exists trees as formulae. in preparation, 2004.

[18] W. Rudin. *Principles of Mathematical analysis*. McGraw-Hill, New York, 1953.

[19] G. Székely. Twin paradox in first-order logical approach, 2003. TDK paper ELTE,(in Hungarian).

[20] K. S. Thorne. *Black holes and time warps*. W. W. Norton and Company, 1994.

[21] C. Wüthrich. On time machines in kerr-newman spacetime, 1999.