Algebras of concepts and their networks

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What is a concept?

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"The center $Z(G)$ of group G" is a concept of groups defined by

$$\varphi(x) \ : \quad \forall y \ (x \cdot y = y \cdot x).$$

"All elements of finite order in G" is not a concept of groups, as

$$\varphi(x) \ : \quad \exists n \ (x^n = e)$$

is not a formula of the language of groups (as the quantifier $\exists n$ ranges over natural numbers, not the elements of the group G).

Meanings of formulas

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The **meaning** *of φ in $\mathcal{M}$* is those sequences of elements of $\mathcal{M}$ that satisfy φ:

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For example, $[\forall y \; (v_0 \cdot y = y \cdot v_0)]^{G} = \{\bar{a} \in G^{\omega} : a_0 \in Z(G)\}$

Concepts form a Booelan algebra

The **concepts** *of structure* $\mathcal{M}$ are subsets of M^ω which are meanings of some formulas:

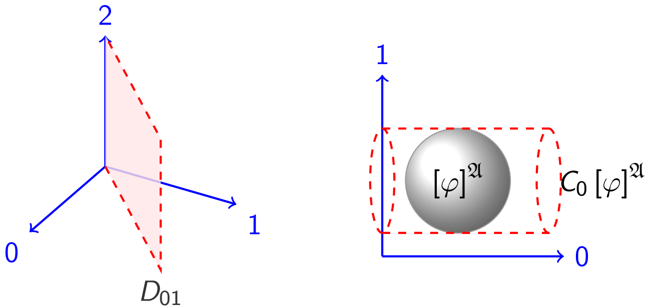
$$\mathsf{Cs}(\mathcal{M}) \stackrel{\text{def}}{=} \left\{ [\varphi]^{\mathcal{M}} : \text{for some formula } \varphi \right\}.$$

Concepts in $\mathsf{Cs}(\mathcal{M})$ are closed under the set theoretical operations intersection, etc.; and they form a Boolean algebra:

$$[\varphi]^{\mathcal{M}} \cap [\psi]^{\mathcal{M}} = [\varphi \& \psi]^{\mathcal{M}}, \quad \text{etc.}$$

Concepts form a Boolean algebra with operators

$$\mathfrak{Cs}(\mathcal{M}) \stackrel{\text{def}}{=} \big(\mathsf{Cs}(\mathcal{M}), \cap, \setminus, C_i, D_{ij}\big)_{i,j<\omega},$$



$$D_{ij} \stackrel{\text{def}}{=} [v_i = v_j]^{\mathcal{M}}$$

$$C_i\,[\varphi]^{\mathcal{M}} \stackrel{\text{def}}{=} [\exists v_i\ \varphi]^{\mathcal{M}}$$

Project on investigating concrete algebras of concepts

Problem (Monk, 2000)

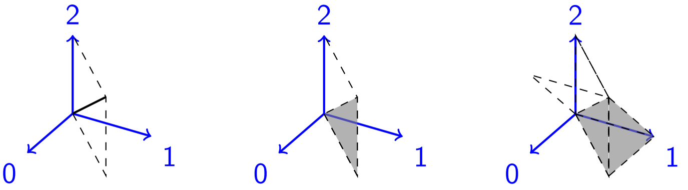
Give a complete description of the concept algebra of $(\mathbb{Q}, <)$.

"A complete description of CsM is known only in the case in which M has only one-place relations. There are many other simple structures where the description of CsM should not be difficult; for example, for M the rationals under their natural ordering." – Don Monk, Logic Journal of the IGPL 8, pp.451–506, (2000)

Theorem (Khaled–Székely, 2020)

Every n-dimensional concept of any dense linear ordering without endpoints $\mathcal{A} = (A, <)$ is the union of some minimal nontrivial n-dimensional concepts (called n-dimensional atoms).

These atom are the non-zero concepts of the following form: for every $i < j < n$, take one of $v_i = v_j$, $v_i < v_j$ or $v_j < v_i$ and put conjunction between these choices.



$[v_0 = v_1 = v_2]^{\mathcal{A}}$ $[v_0 = v_1 < v_2]^{\mathcal{A}}$ $[v_1 < v_0 < v_2]^{\mathcal{A}}$

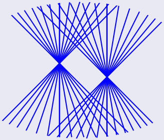
For example, there are exactly 13 atoms from the 27 candidates.

Concept Algebras in Physics

Relativistic and non-relativistic spacetimes

Definition (Relativistic Spacetime $\mathcal{S}r$)

$\mathcal{S}r$ is the system of timelike straight lines:

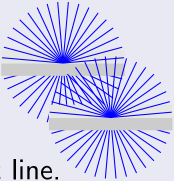


$$\mathcal{S}r = \langle \mathbb{R}^4, \mathrm{col}^t \rangle$$

$\mathrm{col}^t(p,q,r) \iff p,q,r$ are on a timelike straight line.

Definition (Classical non-Relativistic Spacetime $\mathcal{N}t$)

$\mathcal{N}t$ is the system of non-horisontal straight lines:



$$\mathcal{N}t = \langle \mathbb{R}^4, \mathrm{col}^\infty \rangle$$

$\mathrm{col}^\infty(p,q,r) \iff p,q,r$ are on a slanted straight line.

Concept algebra of $\mathcal{S}r$

Theorem (Andréka–Madarász–Németi–Székely, 2019)

- *The unary concepts of $\mathcal{S}r$ are trivial ($\emptyset$ and $\mathbb{R}^4$).*
- *There are exactly 16 binary concepts of $\mathcal{S}r$: they are Boolean combinations of "equal", "timelike related," and "lightlike related."*
- *There are infinitely many ternary concepts of $\mathcal{S}r$. The corresponding ternary concept algebra is atomic.*

Concept algebra of $\mathcal{N}t$

Theorem (Andréa–Madarász–Németi–Székely, 2019)

- *The unary concepts of $\mathcal{N}t$ are trivial ($\emptyset$ and $\mathbb{R}^4$).*
- *There are exactly 8 binary concepts of $\mathcal{N}t$: they are Boolean combinations of "equal", and "being simultaneous."*
- *There are infinitely many ternary concepts of $\mathcal{N}t$. The corresponding ternary concept algebra is atomic.*

Consequences

Corollary

Neither $\mathcal{N}t$ nor $\mathcal{S}r$ can be interpreted into the other.

So using insights gathered by investigating concept algebras $\mathfrak{Cs}(\mathcal{M}_1)$ and $\mathfrak{Cs}(\mathcal{M}_2)$, one can show interesting connections between the corresponding mathematical structures $\mathcal{M}_1$ and $\mathcal{M}_2$.

Let $\mathcal{N}t^{+}$ denote classical spacetime extended with the concept of lightlike conectedness. This structure captures "later" classical kinematics of the nineteenth century at the time of J. C. Maxwell and the search for the "luminiferous ether."

Problem (Andréka, 2017)

Is it true that any concept from $\mathfrak{Cs}(\mathcal{N}t^{+})$ which is not an element of subalgebra $\mathfrak{Cs}(\mathcal{S}r)$ generates together with $\mathfrak{Cs}(\mathcal{S}r)$ the whole $\mathfrak{Cs}(\mathcal{N}t^{+})$?

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If the answer to Andréka's question is positive, then no matter which classical concept we add to special relativity we will get later classical kinematics.

Networks of Concept Algebras

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Definition

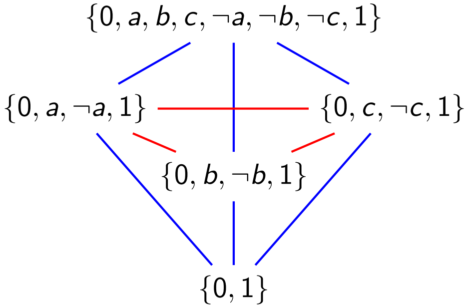
We define the *(generator adding) network of a class of similar algebras* K to be the (possibly infinite) graph whose

- vertices are the members of K,

and which has two types of edges:

- red edges connecting the isomorphic members of K, and
- blue edges connecting any two algebras of K if they are not isomorphic, but one of them is a subalgebra of the other such that the small algebra together with one extra element generates the big algebra.

A small network of Boolean algebras



Generator distance between algebras

Definition

We define the **generator distance** between $\mathfrak{A}$ and $\mathfrak{B}$ in K as:

$$d_{\mathsf{K}}(\mathfrak{A},\mathfrak{B}) \stackrel{\text{def}}{=} \text{the minimum number of blue edges among all the paths from } \mathfrak{A} \text{ to } \mathfrak{B} \text{ in K}$$

if there is a finite path connecting $\mathfrak{A}$ and $\mathfrak{B}$ in K, and

$$d_{\mathsf{K}}(\mathfrak{A},\mathfrak{B}) \stackrel{\text{def}}{=} \infty$$

otherwise.

Example

Let K is be the class of finite dimensional vector spaces over a given field. Then, for all $V, W \in \mathsf{K}$,

$$d_{\mathsf{K}}(V, W) = |dim(V) - dim(W)|.$$

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Theorem (Aslan–Khaled–Székely, 2020)

Let BA *be the class of all Boolean algebras. Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two finite Boolean algebras with* 2^n*-many and* 2^m*-many elements, respectively. Assume that* $n \leq m$*, then*

$$d_{\mathrm{BA}}(\mathfrak{A}, \mathfrak{B}) = \lceil \log_2 m - \log_2 n \rceil.$$

Conceptual distance between theories

theory $\leftrightsquigarrow$ set of formulas

Definition

Conceptual distance $\mathrm{Cd}(T_1, T_2)$ measures the minimal number of concept adding/removing steps that are needed to be taken to reach theory T_2 from T_1.

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Definition

Conceptual distance $\mathrm{Cd}(T_1, T_2)$ measures the minimal number of concept adding/removing steps that are needed to be taken to reach theory T_2 from T_1.

In general, it is rather difficult to determine the concrete conceptual distance between two theories. For example, the main PhD result of K. Lefever states, in terms of conceptual distance, that the conceptual distance between certain axiomatic theories of classical and relativistic kinematics is 1.

Let LT be the class of concept algebras corresponding to theories [i.e., Lindenbaum–Tarski algebras].

Theorem (Aslan–Khaled–Székely, 2020)

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two arbitrary mathematical structures. Then

$$d_{\mathsf{LT}}\big(\mathfrak{Cs}(\mathcal{M}_1), \mathfrak{Cs}(\mathcal{M}_2)\big) = \mathsf{Cd}\big(\mathsf{Th}(\mathcal{M}_1), \mathsf{Th}(\mathcal{M}_2)\big),$$

where $\mathsf{Th}(\mathcal{M}_1)$ *and* $\mathsf{Th}(\mathcal{M}_2)$ *denote the theories of* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

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Problem

How conceptual distance is related to generator distance in the class abstract concept algebras [i.e., cylindric algebras]?

Summary

- Concrete concept algebras are interesting even in themselves.
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Thank you for your attention!