Symmetry Axioms in Relativity Theories

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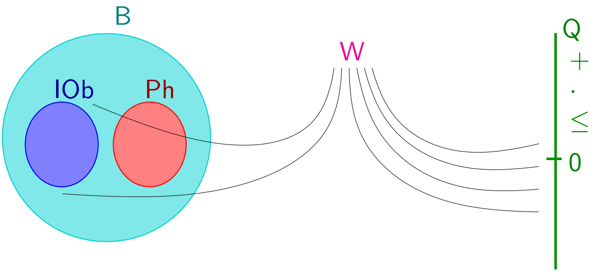
General aims of our school (led by H. Andréka and I. Németi):

- Turn relativity theories to theories of mathematical logic.
- Base relativity theories on simple, unambiguous axioms.
- Demystify relativity theories.
- Make relativity theories modular and easier to change.
- Analyze the logical structure of relativity theories.
- Etc.

A benefit of axiomatization is that we have new questions:

- Which axioms are responsible for a certain theorem?
- How are the possible axioms/axiomatizations related to each other?
- How can these axiomatizations be extended, e.g., towards Quantum Theory?
- How are the independent statements of our axiomatizations related to each other?
- Etc.

Language: $\{ B, IOb, Ph, Q, +, \cdot, \leq, W \}$



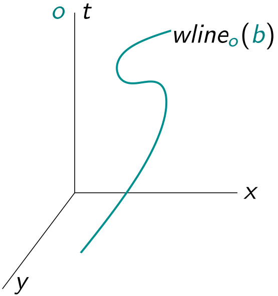
B ⇝ Bodies (things that move)

IOb ⇝ Inertial Observers Ph ⇝ Photons (light signals)

Q ⇝ Quantities $+$, $\cdot$ and $\leq$ ⇝ field operations and ordering

W ⇝ Worldview (a 6-ary relation of type BBQQQQ)

$W(o, b, x, y, z, t) \rightsquigarrow$ "observer o sees (coordinatizes) body b at spacetime location $\langle x, y, z, t \rangle$."



Worldline of body b according to observer o

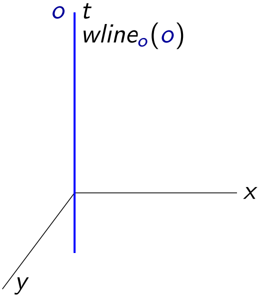
$$wline_o(b) := \{\langle x, y, z, t \rangle \in Q^4 \ : \ W(o, b, x, y, z, t)\}$$

AxField :

The **quantity part** $\langle Q; +, \cdot, \leq \rangle$ *is a Euclidean ordered field.*

AxSelf :

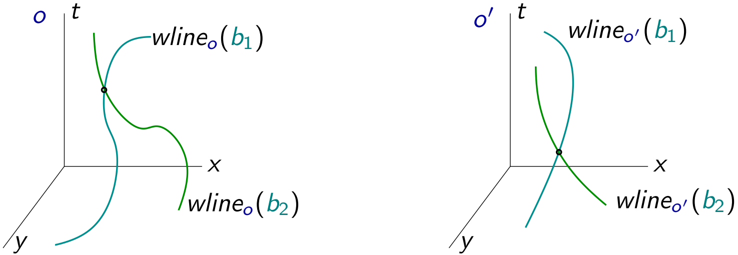
An inertial observer sees himself as standing still at the origin.



$$\forall o, x, y, z, t \;\; \mathsf{IOb}(o) \Longrightarrow \big(\mathsf{W}(o, o, x, y, z, t) \iff x = y = z = 0\big).$$

AxEv :

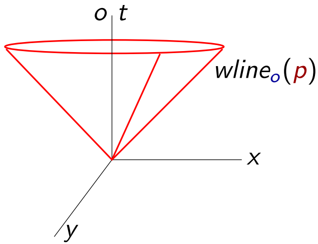
Every inertial observer sees the same events (meetings of bodies).



$$\forall o\, o' x\, y\, z\, t\;\; \mathsf{IOb}(o) \wedge \mathsf{IOb}(o') \Longrightarrow$$
$$\big(\exists x'\, y'\, z'\, t' \;\; \forall b \;\; \mathsf{W}(o,b,x,y,z,t) \iff \mathsf{W}(o',b,x',y',z',t')\big).$$

AxPh :

The speed of light signals is 1 according to any inertial observer.



$$\forall o\, x\, x'\, y\, y'\, z\, z'\, t\, t' \;\; \mathsf{IOb}(o)$$
$$\Longrightarrow \Big(\big(\exists p \quad \mathsf{Ph}(p) \wedge \mathsf{W}(o,p,x,y,z,t) \wedge \mathsf{W}(o,p,x',y',z',t')\big)$$
$$\Longleftrightarrow \; (x'-x)^2 + (y'-y)^2 + (z'-z)^2 = (t'-t)^2 \Big).$$

$$\mathsf{SpecRel}_0 := \{\mathsf{AxField}, \mathsf{AxSelf}, \mathsf{AxEv}, \mathsf{AxPh}\}$$

Theorem

$\mathsf{SpecRel}_0 \models$ *"Worldlines of inertial observers are straight lines."*

Theorem

$\mathsf{SpecRel}_0 \models$ *"No inertial observer can move faster than light."*

Theorem

$\mathsf{SpecRel}_0 \models$ *"Relatively moving inertial observers consider different events simultaneous"*

$$\mathsf{SpecRel}_0 := \{\mathsf{AxField}, \mathsf{AxSelf}, \mathsf{AxEv}, \mathsf{AxPh}\}$$

Theorem

$\mathsf{SpecRel}_0 \models$ "**One of two** *relatively moving inertial observers see that the other's clocks slow down.*"

Theorem

$\mathsf{SpecRel}_0 \models$ "**One of two** *relatively moving (inertial) spaceships shrinks according to the other.*"

AxSymTime :

Any two inertial observers see each others' clocks behaving in the same way.

AxSymDist :

Inertial observers agree as for the spatial distance between events if these events are simultaneous for both of them.

Theorem

$$\mathsf{SpecRel}_0 \not\models \mathsf{AxSymTime} \quad \textit{and} \quad \mathsf{SpecRel}_0 \not\models \mathsf{AxSymDist}$$
$$\mathsf{SpecRel}_0 \models \mathsf{AxSymTime} \iff \mathsf{AxSymDist}$$

$$\mathsf{SpecRel} := \mathsf{SpecRel}_0 + \mathsf{AxSymTime}$$

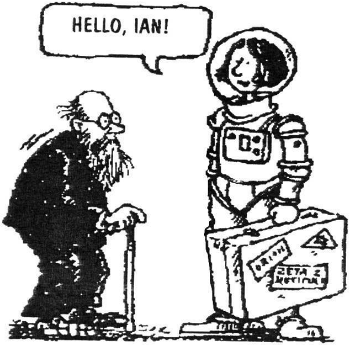
$$\text{SpecRel} := \text{SpecRel}_0 + \text{AxSymTime}$$

Theorem

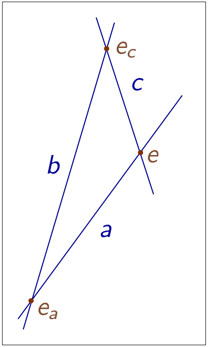
SpecRel $\models$ "*Both of two relatively moving inertial observers see that the other's clocks slow down.*"

Theorem

SpecRel $\models$ "*Both of two relatively moving (inertial) spaceships shrink according to the other.*"



Twin Paradox (TwP) concerns two twin siblings whom we shall call Ann and Ian. Ann travels in a spaceship to some distant star while Ian remains at home. TwP states that when Ann returns home she will be *younger* than her *twin brother* Ian.



TwP :

$$time_b(e_a, e_c) > time_a(e_a, e) + time_c(e, e_c)$$

$$\mathrm{SpecRel} = \{\mathrm{AxField}, \mathrm{AxSelf}, \mathrm{AxEv}, \mathrm{AxPh}, \mathrm{AxSymTime}\}$$
$$\mathrm{SpecRel}_0 = \{\mathrm{AxField}, \mathrm{AxSelf}, \mathrm{AxEv}, \mathrm{AxPh}\}$$

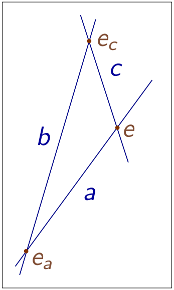
Theorem

$$\mathrm{SpecRel} \models \mathrm{TwP}$$
$$\mathrm{SpecRel}_0 \not\models \mathrm{TwP}$$

How does TwP related to the symmetry axioms?

Is it equivalent to them or is it weaker?

VARIANTS OF TWIN PARADOX



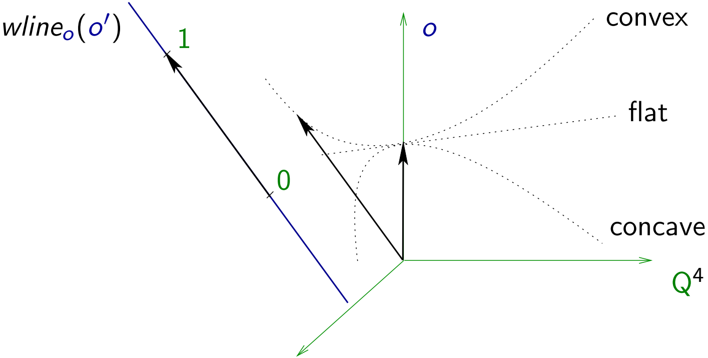
NoTwP :

$$time_b(e_a, e_c) = time_a(e_a, e) + time_c(e, e_c)$$

AntiTwP :

$$time_b(e_a, e_c) < time_a(e_a, e) + time_c(e, e_c)$$

Minkowski Sphere of o



MS_o is the set of time-unit vectors of inertial observers according to o.

Theorem

$\mathrm{SpecRel}_0 \models \mathrm{TwP} \iff MS_o$ *is "convex"*

$\mathrm{SpecRel}_0 \models \mathrm{NoTwP} \iff MS_o$ *is "flat"*

$\mathrm{SpecRel}_0 \models \mathrm{AntiTwP} \iff MS_o$ *is "concave"*

Theorem

$\mathrm{SpecRel}_0 \models \mathrm{AxSymTime} \iff MS_o$ *is a subset of the hyperboloid* $\{\langle x, y, z, t\rangle \in \mathrm{Q}^4 : -x^2 - y^2 - z^2 + t^2 = 1\}$"

Theorem

$\mathrm{SpecRel}_0 \models \mathrm{AxSymTime} \Longrightarrow \mathrm{TwP}$

$\mathrm{SpecRel}_0 \models \mathrm{TwP} \not\Longrightarrow \mathrm{AxSymTime}$

The **worldview transformation** $w_{oo'}$ between observers o and o' relates the coordinate points where o and o' coordinatize the same events, i.e,:

$$w_{oo'}(x,y,z,t : x',y',z',t') \stackrel{def}{\iff} \forall b\ W(o,b,x,y,z,t) \iff W(o',b,x',y',z',t').$$

Theorem

$$\mathsf{SpecRel}_0 \models \forall o\, o'\ \mathsf{IOb}(o) \land \mathsf{IOb}(o') \implies w_{oo'}$$ *"is a Poincaré transformation composed with a dilation and a field-automorphism-induced bijection."*

Theorem

$$\mathsf{SpecRel} \models \forall o\, o'\ \mathsf{IOb}(o) \land \mathsf{IOb}(o') \implies w_{oo'}$$ *"is a Poincaré transformation."*

$$\mathsf{SpecRel} = \{\mathsf{AxField}, \mathsf{AxSelf}, \mathsf{AxEv}, \mathsf{AxPh}, \mathsf{AxSymTime}\}$$

Theorem (Completeness)

SpecRel *is complete with respect to Minkowski spacetimes over Euclidean ordered fields.*

AxSelf^- :

The worldline of an observer is an open interval of the time-axis, in his own worldview.

AxEv^- :

Any observer encounters the events in which he was observed.

AxPh^- :

The instantaneous velocity of photons are 1 in the moment when they are sent out ...

$\mathsf{AxSymTime}^-$:

Any two observers meeting see each others' clocks behaving in the same way at the event of meeting.

AxDiff_n :

The worldview transformations are n-times differentiable functions.

$\mathsf{GenRel}_n := \{\, \mathsf{AxField}, \mathsf{AxSelf}^-, \mathsf{AxEv}^-, \mathsf{AxPh}^-, \mathsf{AxSymTime}^-, \mathsf{AxDiff}_n \,\}$

Theorem (Completeness)

GenRel_n *is complete with respect to the n-times differentiable Lorentzian manifolds over Euclidean ordered fields.*

$\mathsf{GenRel}_\infty := \bigcup_{n\geq 1} \mathsf{GenRel}_n$

Theorem (Completeness)

GenRel_∞ *is complete with respect to the smooth Lorentzian manifolds over Euclidean ordered fields.*

Background materials are available from:
www.renyi.hu/~turms

Thank you for your attention!