A sharpening of Tusnády’s inequality

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Abstract
Let $\varepsilon_1, \ldots, \varepsilon_m$ be i.i.d. random variables with

$$P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2,$$

and $X_m = \sum_{i=1}^m \varepsilon_i$. Let $Y_m$ be a normal random variable with the same first two moments as that of $X_m$. There is a uniquely determined function $\Psi_m$ such that the distribution of $\Psi_m(Y_m)$ equals to the distribution of $X_m$. Tusnády’s inequality states that

$$|\Psi_m(Y_m) - Y_m| \leq \frac{Y_m^2}{m} + 1.$$

Here we propose a sharpened version of this inequality.

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1 Conjecture

Let $\varepsilon_1, \ldots, \varepsilon_m$ be i.i.d. random variables with

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and $X_m = \sum_{i=1}^m \varepsilon_i$. Let $Y_m$ be a normal random variable with the same first two moments as that of $X_m$. Using quantile transformation we can

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see that there is a uniquely determined function $\Psi_m$ such that the distribution of $\Psi_m(Y_m)$ equals to the distribution of $X_m$. The central limit theorem implies that the function $\Psi_m$ is close to the identity for large $m$.

A sharp inequality of Tusnády [12] raised certain interest in the literature ([1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12]).

Let us define the function $f$ on the interval $(0,1)$ as

$$f(x) = \sqrt{(1 + x) \log(1 + x) + (1 - x) \log(1 - x)},$$

set $f(0) = 0$, $f(1) = \sqrt{\log(4)}$. Let us put

$$x_{k,m} = \frac{k - \frac{m}{2}}{\frac{m}{2}}$$

for positive even integers $m$ with $k$ such that $m/2 < k \leq m$, and set

$$p_{k,m} = P(X_m \geq 2k - m) = 2^{-m} \sum_{i=k}^{m} {m \choose i}.$$

Let us define the function $Q$ on the reals as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} du.$$

With those ingredients our conjecture states that

$$Q(\sqrt{m}f(x_{k,m})) < p_{k,m} < Q(\sqrt{m}f(x_{k-1,m}))$$

holds true for $\frac{m}{2} < k \leq m$. Or more sharply

$$2(k-1) - \frac{m}{2} + 0.8964 < mf^{-1}(Q^{-1}(p_{k,m})/\sqrt{m}) < 2(k-1) - \frac{m}{2} + 1.0000 \quad (1)$$

holds true with pessimal parameters $m = k = 10$. It implies that Tusnády’s inequality is sharpened to

$$\left| \Psi_m(Y_m) - mf^{-1}\left(\frac{Y_m}{m}\right) \right| < 1.1036.$$

## 2 Generalization

For an arbitrary random variable $X$ let us consider the function on reals

$$R(t) = Ee^{tX}.$$
restricting ourselves for distributions having finite momentum generators. Next we define

\[ \psi(t) = \frac{R'(t)}{R(t)}, \]

\[ \alpha(x) = t \quad \text{iff} \quad \psi(t) = x, \]

\[ \rho(x) = R(\alpha(x)) \exp(-x\alpha(x)). \]

The probability \( P(\sum_{i=1}^{m} X_i \geq mx) \) is approximately \( \rho(x)^{-m} \) if \( x > EX \). The function \( \rho \) depends on the distribution of \( X \), it is the Chernoff function of \( X \). Let us denote the Chernoff function of the distribution \( F \) of \( X \) by \( \rho_F \), and the corresponding function for standard normal by \( \rho_G \). The quantile transformation between the partial sums of distribution \( F \) with Gaussian ones resemble us to the equation

\[ \rho_F(x) = \rho_G(y) \]

having the property that it gives sharp values for any \( m \). Perhaps the error term is bounded with a bound depending on the distribution of \( X \). For the case symmetrical binomial distribution the error term might be as small as that the quantile curve jumps over its limiting function: it is the informal explanation of our conjecture.

### 3 Numerical Illustration

The function \( \Psi_m \) is shown in Figure 1. called “step” for \( m = 50 \) with a rescaling for random variables

\[ \xi_m = \frac{X_m}{m}, \quad \eta_m = \frac{Y_m}{m}. \]

The function \( f \) is called “limit”, for the sequence of step functions goes to \( f \) after rescaling. The conjecture comes from the observation that the limit function crosses all steps near to their middle. Let us introduce the blow up error term

\[ \Delta_{k,m} = 10 \left( 2k - 1 - m f^{-1} \left( \frac{1}{\sqrt{m}} Q^{-1} \left( \frac{m}{i=0} \left( \begin{array}{c} m \\ i \end{array} \right) 2^{-m} \right) \right) \right), \]

for \( 0 < k \leq m/2 \). In Figure 1. it is labelled as ”Delta”. With these notations \( 0 \) is equivalent with \( 0 < \Delta_{k,m} < 1.036 \). These error terms are shown in Figure 2. for \( 2 \leq m \leq 1000 \). Figure 2. prompts the conjecture that even these curves are convergent. We are a bit perplexed: even the inequality
$0 < \Delta_{1,2} < 1.036$ means that $Q(0.723359) < 0.25 < Q(0.6435214)$. How can we prove such an inequality theoretically?

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**References**


4 Appendix

R- program of Figures 1 and 2.

Q=function(p) -qnorm(p)
G=function(x) (1+x)*log(1+x)+(1-x)*log(1-x))**0.5
Ginv=function(u) {
GG=function(x) G(x)-u
uniroot(GG,c(0,1),f.lower=-u,f.upper=log(4)^.5-u,tol=10^-100)
}

m=50; k=m/2
sum=0; divisor=2**m; bin=
xx=c(1:k+1); yy=c(1:k+1); zz=c(1:k+1);
for (i in 1:k-1){
sum=sum+bin
x=(m-2*i)/m
y=Q(sum/divisor)/(m**.5)
b=Ginv(y)$root
yy[i+1]=y; xx[i+1]=x
bin=(m-i)*bin/(i+1)
zz[i+1]=10*(m-2*i-1-m*b)
xx[k+1]=0; yy[k+1]=0; zz[k+1]=0
kerx=c(0,1.25); kery=c(0,1.15)
plot(kerx, kery, type="n",xlab="eta", ylab="xi",
main="Figure1. Quantile transform, its limit and blownup error, m=50")
for (i in 1:k){
bb=seq(from=yy[i+1], to=yy[i], by=0.01)
cc=bb*0+1; cc=cc*xx[i+1]
points(bb,cc,type="1", col="blue", lwd=2)
cc=seq(from=0, to=0.999, by=0.001)
bb=((1+cc)*log(1+cc)+(1-cc)*log(1-cc))**0.5
points(bb,cc, type="1", col="red", lwd=2)
points(yy,zz, type="1", col="green", lwd=2)
legend(locator(1),c("Limit","Step","Delta"),
lty=c(1,1,1),
col=c("red","blue","green"))
kerx=c(0,1.25); kery=c(0,1.15)
plot(kerx, kery, type="n", xlab="eta", ylab="Delta",
    main="Figure 2. The blownup error")

for (k in 1:500){m=2*k;
  sum=0; divisor=2**m; bin=1
  yy=c(1:k+1); zz=c(1:k+1);
  for (i in 1:k-1){
    sum=sum+bin
    y=Q(sum/divisor)/(m**.5)
    b=Ginv(y)$root
    yy[i+1]=y;
    bin=(m-i)*bin/(i+1)
    zz[i+1]=10*(m-2*i-1-m*b)}
  yy[k+1]=0; zz[k+1]=0
  if (k<100) clr="red" else
    if (k<200) clr="blue" else
      if (k<300) clr="purple" else clr="green"
  points(yy,zz, type="l", col=clr)}
legend(locator(1),c("0<m <= 200","200<m<=400","400<m<=600",
    "600<m<=800","800<m<=1000"),
    lty=c(1,1,1,1,1),
    col=c("red","blue","purple","gray","green"))
Figure 1. Quantile transform, its limit and blown up error, $m=50$

- $\eta$
- $\xi$
- Limit
- Step
- Delta

Graph showing the quantile transform, its limit, and blown up error for $m=50$.
Figure 2. The blownup error