Borel hulls of Haar null sets

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Category arguments

Nowhere differentiable functions

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Level sets

Theorem. (Bruckner, Garg, 1977) For comeager many $f \in C[a,b]$ there exists a countable dense $A \subset (min(f), max(f))$ such that for every $y \in (min(f), max(f)) \setminus A$ the set $f^{-1}(y)$ is perfect and for $y \in A$ the set $f^{-1}(y)$ is a perfect set and an isolated point.

Measure theoretic analogs

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Invariance

Definition. Let $(G,+)$ be a Polish abelian topological group and μ is a Borel measure on G. We say that μ is a *Haar measure* on G if

- for every $t \in G$ and $B \subset G$ Borel $\mu(B) = \mu(t+B)$.
- μ is Borel regular, for every K compact $\mu(K) < \infty$
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Haar measure

Theorem. (Haar, Weil) Let $(G, +)$ be a Polish abelian topological group. There exists a nontrivial Haar measure on G if and only if G is locally compact. Moreover, if μ exists then it is unique up to a multiplicative constant.

Generalization of $\mathcal{N}$

Shy sets

Definition. (Christensen, 1972) Let $(G, +)$ be a Polish abelian group and $S \subset G$. We say that S is *Haar null (shy)* if there exists a universally measurable $U \supset S$ and a continuous Borel probability measure μ on G such that for every $t \in G$ we have $\mu(t + U) = 0$.

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Proposition. Suppose G is locally compact. Then S is Haar null if and only if $\mu(S) = 0$, where μ is the Haar measure on G.

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Relation to Haar measures

Proposition. Suppose G is locally compact. Then S is Haar null if and only if $\mu(S) = 0$, where μ is the Haar measure on G.

Further properties

Proposition. For any Polish abelian group G the Haar null subsets of G form a σ-ideal.

Possible variations

Naive approach

Let $\mathcal{S}_{Naive} = \{A \subset G : (\exists \mu)(\forall t \in G)(\mu(A + t) = 0)\}$.

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Proposition. (CH) If $E \subset (\mathbb{Z}^\omega)^2$ is a well ordering of $\mathbb{Z}^\omega$, then $E \cup E^c = (\mathbb{Z}^\omega)^2$, but E is naively Haar null.

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Negative results

Theorem. (Elekes, Steprans) There exists a non Lebesgue-null $H \subset \mathbb{R}$ and a continuous Borel probability measure μ such that $\forall t \in \mathbb{R}$ we have $\mu(t + H) = 0$.

Possible variations

Definition of shy sets with Γ-hull

Let G be a Polish abelian group, and $\Gamma \subset \mathcal{P}(G)$. We say that a set S is *Haar null with a* Γ*-hull* if

$$(\exists \mu)(\exists H \in \Gamma)(\forall t \in G)(\mu(H + t) = 0) \wedge S \subset H).$$

This family is denoted by $\mathcal{S}_\Gamma$.

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If G locally compact then $\mathcal{S}_{G_\delta} = \mathcal{S}_{\mathcal{UM}}$.
Elekes and Steprans $\Rightarrow$ in $\mathbb{R}$ we have
$\mathcal{S}_{\mathbf{\Pi}^0_2} = \mathcal{S}_{\mathbf{\Delta}^1_1} = \mathcal{S}_{\mathbf{\Sigma}^1_1} = \mathcal{S}_{\mathcal{UM}} \subsetneq \mathcal{S}_{\mathcal{P}(X)}$.

Further known results

Def: $\mathcal{S}_\Gamma = \{S : (\exists \mu)(\exists H \in \Gamma)(\forall t \in G)(\mu(H+t) = 0) \wedge S \subset H)\}$.

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Theorem. (Banakh, 2004) (MA) And G is not locally compact then $cof(\mathcal{S}_{\mathcal{UM}}) > \mathfrak{c}$.
$\Rightarrow \mathcal{S}_{\mathbf{\Delta}^1_1} \neq \mathcal{S}_{\mathcal{UM}}$.

Definability of the counter-examples

Π^1_1 example in L

Theorem. (Z. V.) There exists a Π^1_1 set $\mathcal{H} \subset \mathbb{Z}^\omega$ such that $\mathcal{H}$ is Haar null but there is no $\boldsymbol{\Sigma}^1_1$ Haar null set containing it.

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$\Rightarrow$ enough to prove that every prevalent (co-Haar null) $\mathbf{\Pi}^1_1$ is $\leq_h$-cofinal.

Towards $Con(\mathcal{S}_{\mathbf{\Delta}^1_1} = \mathcal{S}_{\mathbf{\Pi}^1_1})$

Solecki's $\mathcal{S}_{\mathbf{\Delta}^1_1} = \mathcal{S}_{\mathbf{\Sigma}^1_1}$

Theorem. (First reflection) Suppose that X is Polish and $\Phi \subset \mathcal{P}(X)$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. If $A \in \Phi \cap \mathbf{\Sigma}^1_1$ then $\exists B \in \Phi \cap \mathbf{\Delta}^1_1$ such that $A \subset B$.

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Fix a μ measure on a Polish abelian group G and let $c_\mu(A) = \sup\{\mu(A+t) : t \in G\}$, $A \in \Phi_\mu \iff c_\mu(A) = 0$.

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Bounded reflection

Definition. If $\Phi \subset \mathcal{P}(X)$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideal, we say that it satisfies *bounded reflection*, if there exists an ordinal $\gamma < \omega_1$ such that for every $B \in \Phi \cap \mathbf{\Delta}^1_1$ then $\exists D \in \Phi \cap \mathbf{\Pi}^0_\gamma$ with $B \subset D$.

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Preservation of category

Definition. A σ-ideal $\Phi \subset \mathcal{P}(X)$ *preserves category* if whenever $B \subset X \times Y$ is Borel then $\forall^* \forall^\Phi B(x,y) \Rightarrow \forall^\Phi \forall^* B(x,y)$.

Positive result

Theorem. (Clemens, Zapletal) ($\forall x(x^\#$ exists)) Suppose that a σ-ideal Φ preserves category and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Then bounded reflection implies $\mathbf{\Pi}^1_1$-reflection (i.e. $A \in \Phi \cap \mathbf{\Pi}^1_1$ then $\exists B \in \Phi \cap \mathbf{\Delta}^1_1$ such that $A \subset B$.)

Preservation of measure

Theorem?? Suppose that a σ-ideal Φ preserves measure and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Then bounded reflection implies $\mathbf{\Pi}^1_1$-reflection (i.e. $A \in \Phi \cap \mathbf{\Pi}^1_1$ then $\exists B \in \Phi \cap \mathbf{\Delta}^1_1$ such that $A \subset B$.)

Remark

Proposition. For a fixed Borel measure μ the set Φ_μ is a measure preserving $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal.

Corollary

If the previous theorem holds then we have:

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If the previous theorem holds then we have:
Suppose that for every fixed measure μ there exists a $\gamma < \omega_1$ such that every Borel Haar null set with witness μ is contained in a $\mathbf{\Pi}^0_\gamma$ Haar null set with witness $\mu \Rightarrow$
Every $\mathbf{\Pi}^1_1$ Haar null set is contained in a Borel Haar null set.

Towards $Con(\mathcal{S}_{\mathbf{\Delta}^1_1} = \mathcal{S}_{\mathbf{\Pi}^1_1})$

Capacities

Definition. Suppose that X is a Hausdorff space. A *capacity* on X is a map $c : \mathcal{P}(X) \to [0, \infty]$ such that

1. $A \subset B$ implies $c(A) \leq c(B)$
2. $A_0 \subset A_1 \subset \cdots \Rightarrow c(A_n) \to c(\cup A_n)$
3. for any compact $K \subset X$, $c(K) < \infty$ and if $c(K) < r$ then there exists an open $U \subset K$ such that $c(U) < r$.

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Theorem. (Choquet) In a Polish space every $\mathbf{\Sigma}^1_1$ set is c-capacitable for every c capacity.

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Proposition. Let $X = \mathbb{Z}^\omega$. Fix μ, there exists a capacity $\bar{c}_\mu$ such that $\bar{c}_\mu(B) = c_\mu(B) = \sup\{\mu(B+t) : t \in \mathbb{Z}^\omega\}$ for every Borel B.

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Capacitability of $\mathbf{\Pi}^1_1$ sets

Proposition. $\mathbf{\Pi}^1_1$ sets are not universally capacitable.

Questions

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Question. (PD) Does $\mathcal{S}_{G_\delta} = \mathcal{S}_{\mathbf{\Delta}^1_1}$ directly imply $\mathcal{S}_{\mathbf{\Delta}^1_1} = \mathcal{S}_{\mathbf{\Pi}^1_1}$?

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Complementary questions

Question. Is it true that every analytic non-Haar null set contains a Borel non-Haar null set?

Thank you!