Characterisation of order types representable by Baire class 1 functions

Zoltán Vidnyánszky

MTA Rényi Institute

Workshop in set theory, Będlewo, 2014

joint work with Márton Elekes

The original question

Pointwise orderings of functions

Let X be an uncountable Polish space and $\mathcal{F}$ a set of functions $f : X \to \mathbb{R}$.

The original question

Pointwise orderings of functions

Let X be an uncountable Polish space and $\mathcal{F}$ a set of functions $f : X \to \mathbb{R}$.

Definition. For $f, g \in \mathcal{F}$ we say that $f < g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists an $x \in X$ so that $f(x) < g(x)$.

The original question

Pointwise orderings of functions

Let X be an uncountable Polish space and $\mathcal{F}$ a set of functions $f : X \to \mathbb{R}$.

Definition. For $f, g \in \mathcal{F}$ we say that $f < g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists an $x \in X$ so that $f(x) < g(x)$.

General question

Let $(\mathbb{L}, <)$ be an ordering. Does there exist an (order preserving) embedding $(\mathbb{L}, <) \hookrightarrow (\mathcal{F}, <)$?

The original question

Pointwise orderings of functions

Let X be an uncountable Polish space and $\mathcal{F}$ a set of functions $f : X \to \mathbb{R}$.

Definition. For $f, g \in \mathcal{F}$ we say that $f < g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists an $x \in X$ so that $f(x) < g(x)$.

General question

Let $(\mathbb{L}, <)$ be an ordering. Does there exist an (order preserving) embedding $(\mathbb{L}, <) \hookrightarrow (\mathcal{F}, <)$? Terminology: we also say that $\mathbb{L}$ is representable in $\mathcal{F}$.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.
In fact, there exist embeddings $(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$ and $([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.
In fact, there exist embeddings $(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$ and $([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$.

The proof

$([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$ is trivial.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.
In fact, there exist embeddings $(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$ and $([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$.

The proof

$([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$ is trivial.
$(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$: The set of closed sets of a Polish space Y (denoted by $\mathbf{\Pi}^0_1(Y)$) forms a poset with the strict inclusion.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.
In fact, there exist embeddings $(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$ and $([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$.

The proof

$([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$ is trivial.
$(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$: The set of closed sets of a Polish space Y (denoted by $\mathbf{\Pi}^0_1(Y)$) forms a poset with the strict inclusion.
Clearly, the map $f \mapsto \text{subgraph}(f) = \{(x, y) : y \leq f(x)\}$ is an embedding $(C(X, \mathbb{R}), <) \hookrightarrow (\mathbf{\Pi}^0_1(X \times \mathbb{R}), \subset)$.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.
In fact, there exist embeddings $(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$ and $([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$.

The proof

$([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$ is trivial.
$(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$: The set of closed sets of a Polish space Y (denoted by $\mathbf{\Pi}^0_1(Y)$) forms a poset with the strict inclusion.
Clearly, the map $f \mapsto \text{subgraph}(f) = \{(x, y) : y \leq f(x)\}$ is an embedding $(C(X, \mathbb{R}), <) \hookrightarrow (\mathbf{\Pi}^0_1(X \times \mathbb{R}), \subset)$.
Now let $\{U_n : n \in \omega\}$ be a basis of $X \times \mathbb{R}$.

Known results

Continuous case

Theorem. (Folklore) If $\mathcal{F} = C(X, \mathbb{R})$ then $(\mathbb{L}, <)$ representable in $(\mathcal{F}, <)$ if and only if it is embeddable into $([0,1], <)$.
In fact, there exist embeddings $(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$ and $([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$.

The proof

$([0,1], <) \hookrightarrow (C(X, \mathbb{R}), <)$ is trivial.
$(C(X, \mathbb{R}), <) \hookrightarrow ([0,1], <)$: The set of closed sets of a Polish space Y (denoted by $\mathbf{\Pi}^0_1(Y)$) forms a poset with the strict inclusion.
Clearly, the map $f \mapsto \text{subgraph}(f) = \{(x, y) : y \leq f(x)\}$ is an embedding $(C(X, \mathbb{R}), <) \hookrightarrow (\mathbf{\Pi}^0_1(X \times \mathbb{R}), \subset)$.
Now let $\{U_n : n \in \omega\}$ be a basis of $X \times \mathbb{R}$.
Map $F \in \mathbf{\Pi}^0_1(X \times \mathbb{R})$ to $\sum_{U_n \cap F \neq \emptyset} 3^{-n-1}$.

Known results

Observe that we did not use the continuity, just that the sets subgraph(f) are closed.

Known results

Observe that we did not use the continuity, just that the sets subgraph(f) are closed.

Baire class 1 functions

Definition. A function $f : X \to \mathbb{R}$ is *Baire class 1* if it is the pointwise limit of continuous functions. Notation: $\mathcal{B}_1(X)$.

Known results

Observe that we did not use the continuity, just that the sets subgraph(f) are closed.

Baire class 1 functions

Definition. A function $f : X \to \mathbb{R}$ is *Baire class 1* if it is the pointwise limit of continuous functions. Notation: $\mathcal{B}_1(X)$.

Kuratowski's theorem

Theorem. (Kuratowski, 60s) ω_1 and ω_1^* are not representable in $(\mathcal{B}_1(X), <)$.

Known results

Observe that we did not use the continuity, just that the sets subgraph(f) are closed.

Baire class 1 functions

Definition. A function $f : X \to \mathbb{R}$ is *Baire class 1* if it is the pointwise limit of continuous functions. Notation: $\mathcal{B}_1(X)$.

Kuratowski's theorem

Theorem. (Kuratowski, 60s) ω_1 and ω_1^* are not representable in $(\mathcal{B}_1(X), <)$.

Is this a characterisation?

Theorem. (Komjáth, 1990) Consistently no:

Known results

Observe that we did not use the continuity, just that the sets subgraph(f) are closed.

Baire class 1 functions

Definition. A function $f : X \to \mathbb{R}$ is *Baire class 1* if it is the pointwise limit of continuous functions. Notation: $\mathcal{B}_1(X)$.

Kuratowski's theorem

Theorem. (Kuratowski, 60s) ω_1 and ω_1^* are not representable in $(\mathcal{B}_1(X), <)$.

Is this a characterisation?

Theorem. (Komjáth, 1990) Consistently no: If $(\mathbb{S}, <)$ is a Suslin line, then $(\mathbb{S}, <) \not\hookrightarrow (\mathcal{B}_1(X), <)$.

Known results

A non-characterisation result

Theorem. (Elekes, Steprāns, 2006) There exists a linear ordering $(\mathbb{L}, <)$ so that neither ω_1 nor ω_1^* is embeddable into $\mathbb{L}$, but $(\mathbb{L}, <) \not\hookrightarrow (\mathcal{B}_1(X), <)$.

Known results

A non-characterisation result

Theorem. (Elekes, Steprāns, 2006) There exists a linear ordering $(\mathbb{L}, <)$ so that neither ω_1 nor ω_1^* is embeddable into $\mathbb{L}$, but $(\mathbb{L}, <) \not\hookrightarrow (\mathcal{B}_1(X), <)$.

The positive direction

Theorem. (Elekes, Steprāns, 2006) (MA) If $|\mathbb{L}| < \mathfrak{c}$ and neither ω_1 nor ω_1^* is embeddable into $(\mathbb{L}, <)$ then $(\mathbb{L}, <) \hookrightarrow (\mathcal{B}_1(X), <)$.

Known results

A non-characterisation result

Theorem. (Elekes, Steprāns, 2006) There exists a linear ordering $(\mathbb{L}, <)$ so that neither ω_1 nor ω_1^* is embeddable into $\mathbb{L}$, but $(\mathbb{L}, <) \not\hookrightarrow (\mathcal{B}_1(X), <)$.

The positive direction

Theorem. (Elekes, Steprāns, 2006) (MA) If $|\mathbb{L}| < \mathfrak{c}$ and neither ω_1 nor ω_1^* is embeddable into $(\mathbb{L}, <)$ then $(\mathbb{L}, <) \hookrightarrow (\mathcal{B}_1(X), <)$.

Remark on Baire class α

Theorem. (Komjáth, 1990) If $\alpha > 1$ the existence of $\omega_2 \hookrightarrow (\mathcal{B}_\alpha(X), <)$ is already independent of ZFC.

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main Theorem. (Elekes, V.) There exists a universal linear ordering representable by Baire class 1 functions.

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main Theorem. (Elekes, V.) There exists a universal linear ordering representable by Baire class 1 functions.

The universal ordering: $([0,1]^{<\omega_1}_{sd}, <_{altlex})$

We denote the set of *strictly* monotone decreasing transfinite sequences of reals in $[0,1]$ with last element 0 by $[0,1]^{<\omega_1}_{sd}$.

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main Theorem. (Elekes, V.) There exists a universal linear ordering representable by Baire class 1 functions.

The universal ordering: $([0,1]^{<\omega_1}_{sd}, <_{altlex})$

We denote the set of *strictly* monotone decreasing transfinite sequences of reals in $[0,1]$ with last element 0 by $[0,1]^{<\omega_1}_{sd}$.
Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{sd}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main Theorem. (Elekes, V.) There exists a universal linear ordering representable by Baire class 1 functions.

The universal ordering: $([0,1]^{<\omega_1}_{sd}, <_{altlex})$

We denote the set of *strictly* monotone decreasing transfinite sequences of reals in $[0,1]$ with last element 0 by $[0,1]^{<\omega_1}_{sd}$.
Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{sd}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

$$x_\delta < x'_\delta \text{ if } \delta \text{ is even or}$$

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main Theorem. (Elekes, V.) There exists a universal linear ordering representable by Baire class 1 functions.

The universal ordering: $([0,1]^{<\omega_1}_{sd}, <_{altlex})$

We denote the set of *strictly* monotone decreasing transfinite sequences of reals in $[0,1]$ with last element 0 by $[0,1]^{<\omega_1}_{sd}$.
Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{sd}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

$$x_\delta < x'_\delta \text{ if } \delta \text{ is even or } x_\delta > x'_\delta \text{ if } \delta \text{ is odd.}$$

Main question

Question. (Laczkovich, 1984) Which are the linear orderings representable by Baire class 1 functions?

Main Theorem. (Elekes, V.) There exists a universal linear ordering representable by Baire class 1 functions.

The universal ordering: $([0,1]^{<\omega_1}_{sd}, <_{altlex})$

We denote the set of *strictly* monotone decreasing transfinite sequences of reals in $[0,1]$ with last element 0 by $[0,1]^{<\omega_1}_{sd}$.
Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{sd}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

$$x_\delta < x'_\delta \text{ if } \delta \text{ is even or } x_\delta > x'_\delta \text{ if } \delta \text{ is odd.}$$

In fact, there exist $(\mathcal{B}_1(X), <) \hookrightarrow ([0,1]^{<\omega_1}_{sd}, <_{altlex})$ and $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathcal{B}_1(X), <)$.

Ambiguous sets

Definition. A set $A \subset X$ is called ambiguous if it is F_σ and G_δ. The collection of ambigous subsets of X is denoted by $\mathbf{\Delta}^0_2(X)$.

Ambiguous sets

Definition. A set $A \subset X$ is called ambiguous if it is F_σ and G_δ. The collection of ambigous subsets of X is denoted by $\mathbf{\Delta}^0_2(X)$.

Remark

A characteristic function χ_A is Baire-1 if and only if $A \in \mathbf{\Delta}^0_2$.

Ambiguous sets

Definition. A set $A \subset X$ is called ambiguous if it is F_σ and G_δ. The collection of ambigous subsets of X is denoted by $\mathbf{\Delta}^0_2(X)$.

Remark

A characteristic function χ_A is Baire-1 if and only if $A \in \mathbf{\Delta}^0_2$. However, for a Baire-1 function f the sets $\{(x,y) : y \leq f(x)\}$ and $\{(x,y) : y < f(x)\}$ are typically not ambigous.

Theorem. (Hausdorff, Kuratowski) A set A is $\mathbf{\Delta}^0_2$ if and only if there exists a strictly decreasing continuous transfinite sequence of closed sets $(F_\beta)_{\beta<\alpha}$ for some $\alpha < \omega_1$ so that

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n \in \omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Theorem. (Hausdorff, Kuratowski) A set A is $\mathbf{\Delta}^0_2$ if and only if there exists a strictly decreasing continuous transfinite sequence of closed sets $(F_\beta)_{\beta<\alpha}$ for some $\alpha < \omega_1$ so that

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n \in \omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Let $A \subset X$ be arbitrary and $F \subset X$ closed.

Theorem. (Hausdorff, Kuratowski) A set A is $\mathbf{\Delta}^0_2$ if and only if there exists a strictly decreasing continuous transfinite sequence of closed sets $(F_\beta)_{\beta<\alpha}$ for some $\alpha < \omega_1$ so that

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n \in \omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Let $A \subset X$ be arbitrary and $F \subset X$ closed. Let $\partial_F(A)$ be

$$\overline{A \cap F} \cap \overline{A^c \cap F} \text{ (= the boundary of } A \text{ in } F).$$

Theorem. (Hausdorff, Kuratowski) A set A is $\mathbf{\Delta}^0_2$ if and only if there exists a strictly decreasing continuous transfinite sequence of closed sets $(F_\beta)_{\beta<\alpha}$ for some $\alpha < \omega_1$ so that

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n \in \omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Let $A \subset X$ be arbitrary and $F \subset X$ closed. Let $\partial_F(A)$ be

$$\overline{A \cap F} \cap \overline{A^c \cap F} \text{ (= the boundary of } A \text{ in } F).$$

Now let $F_0 = X$ and define for γ, γ' limit and $n \in \omega$ the closed set $F_{\gamma+n}$ by induction:

$$F_{\gamma+2n+2} = \partial_{F_{\gamma+2n}}(A),$$

Theorem. (Hausdorff, Kuratowski) A set A is $\mathbf{\Delta}^0_2$ if and only if there exists a strictly decreasing continuous transfinite sequence of closed sets $(F_\beta)_{\beta<\alpha}$ for some $\alpha < \omega_1$ so that

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n \in \omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Let $A \subset X$ be arbitrary and $F \subset X$ closed. Let $\partial_F(A)$ be

$$\overline{A \cap F} \cap \overline{A^c \cap F} \ (= \text{ the boundary of } A \text{ in } F).$$

Now let $F_0 = X$ and define for γ, γ' limit and $n \in \omega$ the closed set $F_{\gamma+n}$ by induction:

$$F_{\gamma+2n+2} = \partial_{F_{\gamma+2n}}(A), \ F_{\gamma+2n+1} = \overline{A^c \cap F_{\gamma+2n}},$$

Theorem. (Hausdorff, Kuratowski) A set A is $\mathbf{\Delta}^0_2$ if and only if there exists a strictly decreasing continuous transfinite sequence of closed sets $(F_\beta)_{\beta<\alpha}$ for some $\alpha < \omega_1$ so that

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n \in \omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Let $A \subset X$ be arbitrary and $F \subset X$ closed. Let $\partial_F(A)$ be

$$\overline{A \cap F} \cap \overline{A^c \cap F} \ (= \text{ the boundary of } A \text{ in } F).$$

Now let $F_0 = X$ and define for γ, γ' limit and $n \in \omega$ the closed set $F_{\gamma+n}$ by induction:

$$F_{\gamma+2n+2} = \partial_{F_{\gamma+2n}}(A), \ F_{\gamma+2n+1} = \overline{A^c \cap F_{\gamma+2n}},$$

$$F_\gamma = \bigcap_{\gamma'+2n<\gamma} F_{\gamma'+2n}.$$

Proposition. (Elekes, V.) There exists a function $\Psi : \mathbf{\Delta}^0_2(X) \to \mathbf{\Pi}^0_1(X)^{<\omega_1}$ with $\Psi(A) = (F_\beta)_{\beta<\alpha}$ with the following properties:

1. $(F_\beta)_{\beta<\alpha}$ is strictly decreasing and

$$A = \bigcup_{\substack{\gamma<\alpha, \gamma\in Lim \\ n\in\omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$

Proposition. (Elekes, V.) There exists a function $\Psi : \mathbf{\Delta}^0_2(X) \to \mathbf{\Pi}^0_1(X)^{<\omega_1}$ with $\Psi(A) = (F_\beta)_{\beta<\alpha}$ with the following properties:

1. $(F_\beta)_{\beta<\alpha}$ is strictly decreasing and
$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n\in\omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$
2. (Weak preservation of inclusion) If $A \subsetneq A'$ and $\Psi(A') = (F'_\beta)_{\beta<\alpha'}$ and δ is minimal so that $F_\delta \neq F'_\delta$ then
$$F_\delta \subsetneq F'_\delta \text{ if } \delta \text{ is even}$$
and

Proposition. (Elekes, V.) There exists a function $\Psi : \mathbf{\Delta}^0_2(X) \to \mathbf{\Pi}^0_1(X)^{<\omega_1}$ with $\Psi(A) = (F_\beta)_{\beta<\alpha}$ with the following properties:

1. $(F_\beta)_{\beta<\alpha}$ is strictly decreasing and
$$A = \bigcup_{\substack{\gamma<\alpha, \gamma \in Lim \\ n\in\omega}} (F_{\gamma+2n} \setminus F_{\gamma+2n+1}).$$
2. (Weak preservation of inclusion) If $A \subsetneqq A'$ and $\Psi(A') = (F'_\beta)_{\beta<\alpha'}$ and δ is minimal so that $F_\delta \neq F'_\delta$ then
$$F_\delta \subsetneqq F'_\delta \text{ if } \delta \text{ is even}$$
and
$$F_\delta \supsetneqq F'_\delta \text{ if } \delta \text{ is odd.}$$

$(\mathcal{B}_1(X), <) \hookrightarrow ([0,1]^{<\omega_1}_{sd}, <_{altlex})$

Recall the definition of the universal ordering:
We denote the set of *strictly* monotone decreasing continuous transfinite sequences of reals in $[0,1]$ by $[0,1]^{<\omega_1}_{sd}$.
Let $\bar{x} = (x_\beta)_{\beta<\alpha}, \bar{x}' = (x'_\beta)_{\beta<\alpha'} \in [0,1]^{<\omega_1}_{sd}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

$$x_\delta < x'_\delta \text{ if } \delta \text{ is even or } x_\delta > x'_\delta \text{ if } \delta \text{ is odd.}$$

Recall the definition of the universal ordering:
We denote the set of *strictly* monotone decreasing continuous transfinite sequences of reals in $[0,1]$ by $[0,1]^{<\omega_1}_{sd}$.
Let $\bar{x} = (x_\beta)_{\beta<\alpha}, \bar{x}' = (x'_\beta)_{\beta<\alpha'} \in [0,1]^{<\omega_1}_{sd}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

$$x_\delta < x'_\delta \text{ if } \delta \text{ is even or } x_\delta > x'_\delta \text{ if } \delta \text{ is odd.}$$

Using the embedding $(\mathbf{\Pi}^0_1(X), \subset) \hookrightarrow ([0,1], <)$ we obtain:

Recall the definition of the universal ordering:
We denote the set of *strictly* monotone decreasing continuous transfinite sequences of reals in $[0,1]$ by $[0,1]_{sd}^{<\omega_1}$.
Let $\bar{x} = (x_\beta)_{\beta<\alpha}, \bar{x}' = (x'_\beta)_{\beta<\alpha'} \in [0,1]_{sd}^{<\omega_1}$ and let δ be minimal so that $x_\delta \neq x'_\delta$. We say that $\bar{x} <_{altlex} \bar{x}' \iff$

$$x_\delta < x'_\delta \text{ if } \delta \text{ is even or } x_\delta > x'_\delta \text{ if } \delta \text{ is odd.}$$

Using the embedding $(\mathbf{\Pi}^0_1(X), \subset) \hookrightarrow ([0,1], <)$ we obtain:

Concluding result

Theorem. $(\mathbf{\Delta}^0_2(X), \subset) \hookrightarrow ([0,1]_{sd}^{\omega_1}, <_{altlex})$.

Hausdorff analysis for Baire class 1 functions

Theorem. (Kechris, Louveau, 1990) Suppose that f is a bounded nonnegative Baire class 1 function. Then there exists a transfinite, strictly decreasing sequence of nonnegative, upper semi-continuous functions $(f_\beta)_{\beta<\alpha}$ so that

$$f = \sum_{\beta<\alpha}{}^{*} (-1)^\beta f_\beta.$$

Where $\sum^*$ is the generalized alternating sum.

Hausdorff analysis for Baire class 1 functions

Theorem. (Kechris, Louveau, 1990) Suppose that f is a bounded nonnegative Baire class 1 function. Then there exists a transfinite, strictly decreasing sequence of nonnegative, upper semi-continuous functions $(f_\beta)_{\beta<\alpha}$ so that

$$f = \sum_{\beta<\alpha}{}^{*} (-1)^\beta f_\beta.$$

Where $\sum^*$ is the generalized alternating sum.

Embedding for Baire class 1

Theorem. $(\mathcal{B}_1(X), <) \hookrightarrow ([0,1]^{<\omega_1}_{sd}, <_{altlex})$.

The other direction: $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathcal{B}_1(X), <)$

Theorem. (Elekes, V.) The converse is also true, in fact $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$.

Theorem. (Elekes, V.) The converse is also true, in fact $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$.

About the proof

For X and X' uncountable σ-compact spaces it was proved by Elekes that $(\mathcal{B}_1(X), <) \hookrightarrow (\mathcal{B}_1(X'), <)$.

The other direction: $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathcal{B}_1(X), <)$

Theorem. (Elekes, V.) The converse is also true, in fact $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$.

About the proof

For X and X' uncountable σ-compact spaces it was proved by Elekes that $(\mathcal{B}_1(X), <) \hookrightarrow (\mathcal{B}_1(X'), <)$.
So it was enough to prove that $([0,1]^{<\omega_1}_{sd}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(\mathcal{K}([0,1]^2)), \subset)$.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.
- Komjáth: a forcing-free proof of the non-representability of Suslin lines.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.
- Komjáth: a forcing-free proof of the non-representability of Suslin lines.
- The linear orders representable by Baire class 1 functions are the same in all Polish spaces.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.
- Komjáth: a forcing-free proof of the non-representability of Suslin lines.
- The linear orders representable by Baire class 1 functions are the same in all Polish spaces.
- Every linearly ordered set which is representable is also representable by characteristic functions,

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.
- Komjáth: a forcing-free proof of the non-representability of Suslin lines.
- The linear orders representable by Baire class 1 functions are the same in all Polish spaces.
- Every linearly ordered set which is representable is also representable by characteristic functions, in fact $(\mathcal{B}_1(X), <) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.
- Komjáth: a forcing-free proof of the non-representability of Suslin lines.
- The linear orders representable by Baire class 1 functions are the same in all Polish spaces.
- Every linearly ordered set which is representable is also representable by characteristic functions, in fact $(\mathcal{B}_1(X), <) \hookrightarrow (\mathbf{\Delta}_2^0(X), \subset)$.
- Lexicographical countable products of representable linearly ordered sets are also representable.

Applications

- Kuratowski: ω_1 and ω_1^* are not representable.
- Elekes-Steprāns: under MA every order of cardinality less then $\mathfrak{c}$ is representable if and only if ω_1 or ω_1^* is not embeddable into it.
- Komjáth: a forcing-free proof of the non-representability of Suslin lines.
- The linear orders representable by Baire class 1 functions are the same in all Polish spaces.
- Every linearly ordered set which is representable is also representable by characteristic functions, in fact $(\mathcal{B}_1(X), <) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$.
- Lexicographical countable products of representable linearly ordered sets are also representable.
- Completions of a representable linearly ordered sets are not necessarily representable.

Open problems

Question. What can we say about linear orderings representable in higher Baire classes in terms of universal orderings? What if we consider the poset $(\mathbf{\Sigma}^0_\alpha(X), \subset)$ for some $\alpha \geq 2$?

Open problems

Question. What can we say about linear orderings representable in higher Baire classes in terms of universal orderings? What if we consider the poset $(\mathbf{\Sigma}^0_\alpha(X), \subset)$ for some $\alpha \geq 2$?

Question. Does there exist an embedding $(\mathcal{B}_1(X), <) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$ so that $(\mathcal{B}_1(X), <)$ is (as a poset) isomorphic to its image?

Open problems

Question. What can we say about linear orderings representable in higher Baire classes in terms of universal orderings? What if we consider the poset $(\mathbf{\Sigma}^0_\alpha(X), \subset)$ for some $\alpha \geq 2$?

Question. Does there exist an embedding $(\mathcal{B}_1(X), <) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subset)$ so that $(\mathcal{B}_1(X), <)$ is (as a poset) isomorphic to its image?

Question. Does there exist a universal linearly ordered set if X is only separable metrizable?

Thank you for your attention!