Ranks on Baire class ξ functions

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Joint work with Márton Elekes and Zoltán Vidnyánszky.

Definition and motivation

Let us denote the class of Baire-ξ functions by $\mathcal{B}_\xi$. By a rank on $\mathcal{B}_\xi$ we mean a map $\mathcal{B}_\xi \to \omega_1$.

Kechris and Louveau investigated the case of Baire class 1 functions. Our aim is to generalise their results to Baire class ξ functions.

Question

Are there ranks on the class of Baire-ξ functions with nice properties?

Some of the important properties are:

- unboundedness in ω_1
- "linearity" i.e. $\mathrm{rk}(cf + g) \leq \max\{\mathrm{rk}(f), \mathrm{rk}(g)\} \cdot \omega$
- translation invariance (if the space is a Polish group)

First of all, let us show an example of the ranks already defined on the Baire-1 functions.

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Ranks on Baire class 1 functions

It is well-known that for every $H \in \Delta^0_2$ there exists a decreasing continuous sequence of closed sets $\{F_\eta\}_{\eta<\xi}$ such that

$$H = \bigcup_{\substack{\eta<\xi \\ \eta \text{ is even}}} F_\eta \setminus F_{\eta+1}.$$

Define $\mathrm{rk}_{\Delta^0_2}(H)$ as the minimal length of such a sequence.

For $A, B \subset X$ disjoint G_δ sets define

$$\alpha(A,B) = \min\{\mathrm{rk}_{\Delta^0_2}(H) : A \subset H, B \subset H^c, H \in \Delta^0_2\}.$$

Finally,

Definition

The **separation rank** of a Baire class 1 function $f : X \to \mathbb{R}$ is

$$\alpha(f) = \sup_{\substack{p,q\in\mathbb{Q} \\ p<q}} \alpha\left((f \le p), (f \ge q)\right).$$

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The other two ranks are the **oscillation rank** ($\beta(f)$) and the **convergence rank** ($\gamma(f)$), but we will not need the exact form of the somewhat technical definitions, so we omit them.

However, it is interesting to note that these three ranks are defined according to the three classical characterisations of the Baire class 1 functions.

Baire-1 ranks

separation rank (α) $\longleftrightarrow$ level sets of the function
oscillation rank (β) $\longleftrightarrow$ continuity point restricted to every nonempty closed set
convergence rank (γ) $\longleftrightarrow$ pointwise limit of continuous functions

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These ranks more or less satisfy the properties we want.

- all three ranks are unbounded in ω_1 (it can be shown for α and $\alpha(f) \leq \beta(f) \leq \gamma(f)$ modulo some finite difference)
- β and γ are "linear", i.e.

$$\beta(cf + g) \leq \max\{\beta(f), \beta(g)\} \cdot \omega$$

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Ranks on Baire class ξ functions

The most obvious thing to do is to generalize one of the ranks on Baire-1 functions. We can only do that with the separation rank.

Replace $\mathbf{\Delta}^0_2$, closed and G_δ sets by $\mathbf{\Delta}^0_{\xi+1}$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Pi}^0_{\xi+1}$ sets in the definition of the separation rank.

Let us denote this rank by

$$\alpha_\xi.$$

It turns out that this rank is unbounded, but is not linear, since

$$\sup\{\alpha_\xi(f+g) : f, g \in \mathcal{B}_\xi, \alpha_\xi(f) = \alpha_\xi(g) = 2\} = \omega_1.$$

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The natural remedy is the following.

Definition

For $f \in \mathcal{B}_\xi$ let

$$\overline{\alpha_\xi}(f) = \min\{\max\{\alpha_\xi(f_1), \dots, \alpha_\xi(f_n)\} : n \in \omega, f_1, \dots, f_n \in \mathcal{B}_\xi, f = f_1 + \cdots + f_n\}.$$

This time we only have a partial result.

Theorem (Elekes-K-Vidnyánszky)

*$\overline{\alpha_\xi}$ is bounded on the **characteristic** Baire class ξ functions.*

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Limit ranks

Another way to construct $\mathcal{B}_\xi$ ranks is the following.

Definition

For a rank ρ on $\mathcal{B}_1$ and $f \in \mathcal{B}_2$ let

$$\rho'_2(f) = \min_{\substack{f_n \to f \\ f_n \in \mathcal{B}_1}} \sup_n \rho(f_n).$$

However, to our great surprise, this does not work.

Theorem (Elekes-K-Vidnyánszky)

The ranks α'_2, β'_2 and γ'_2 are all bounded in ω_1! Actually, bounded by ω.

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Topology refinement

A bit less natural approach involves topology refinements.

For an $f \in \mathcal{B}_\xi$ let

$$T_{f,\xi} = \{\tau' : \tau' \text{ is Polish}, f \in \mathcal{B}_1(\tau'), \tau' \subset \mathbf{\Sigma}^0_\xi(\tau)\}.$$

And now the definition of the rank is the following.

Definition

For a rank ρ on $\mathcal{B}_1$ and $f \in \mathcal{B}_\xi$ let

$$\rho^*_\xi(f) = \inf_{\tau' \in T_{f,\xi}} \rho_{\tau'}(f),$$

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Topology refinement

This turns out to be good for our purposes.

Theorem (Elekes-K-Vidnyánszky)

The ranks β^*_ξ and γ^*_ξ are nice.

An application to the solvability of systems of difference equations

Definition

$D : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is said to be a **difference operator**, if there are real number a_i, b_i $(i = 1, \dots, n)$ such that for every $f \in \mathbb{R}^{\mathbb{R}}$

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i).$$

Definition

A system of difference equations is

$$D_i(f) = g_i \quad (i \in I),$$

where I is an arbitrary set of indices D_i is a difference operator and g_i is a given function for every $i \in I$.

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Elekes and Laczkovich investigated the solvability cardinal of classes of functions.

Definition (Elekes-Laczkovich)

Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be a class of real functions. The solvability cardinal of $\mathcal{F}$ is the minimal cardinal $sc(\mathcal{F})$ with the property that if every subsystem of size less than $sc(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$, then the whole system has a solution in $\mathcal{F}$.

Corollary (Elekes-K-Vidnyánszky)

If $\xi \geq 2$, then $sc(\{f : f \in \mathcal{B}_\xi\}) \geq \omega_2$.

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