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Descriptive set theoretical methods and their applications

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Contents

Chapter 1

Introduction

Descriptive set theory is a branch of mathematics studying the structure of subsets of the real line, or in general, *Polish spaces* (separable, completely metrisable topological spaces). It has a wide range of applications from functional analysis through the theory of group actions to model theory.

Descriptive set theory classifies objects by their complexity and forms hierarchies accordingly. The most fundamental classification is the so called *Borel hierarchy.* Let X be a Polish space. A subset of X is called *Borel* if it is an element of the smallest σ-algebra containing all open subsets of X. The collection of Borel subsets of X is denoted by $\mathbf{\Delta}^1_1(X)$. Borel sets naturally form a hierarchy that can be defined inductively:

$$\mathbf{\Sigma}^0_1(X) \text{ is the set of open sets,}$$

if $\xi < \omega_1$ is an ordinal then

$$\mathbf{\Pi}^0_\xi(X) \text{ is the set of the complements of sets in } \mathbf{\Sigma}^0_\xi(X)$$

and

$$\mathbf{\Sigma}^0_\xi(X) = \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\xi_n}(X) \text{ for some } \xi_n < \xi\}.$$

The sets $\mathbf{\Sigma}^0_\xi(X)$ and $\mathbf{\Pi}^0_\xi(X)$ are called the *ξth additive and multiplicative classes of the Borel hierarchy*, respectively. It is well known that the collection of the subsets of an uncountable Polish space is much larger than the class of the Borel sets. For example, the continuous image of a Borel set is not necessarily Borel. Continuous images of Borel sets are called *analytic*, their complements are called *coanalytic* sets, denoted by $\mathbf{\Sigma}^1_1(X)$ and $\mathbf{\Pi}^1_1(X)$. These are the first two classes of the so called *projective hierarchy.*

Another very important classification is the *Baire hierarchy* of functions. Let us denote by $\mathcal{B}_0(X)$ the family of real valued continuous functions on X. The Baire classes are also defined inductively, as follows. Let $\xi < \omega_1$ be an ordinal. A function is called a *Baire class ξ* function (i. e. it is the element of $\mathcal{B}_\xi(X)$) if it is the pointwise limit of functions that are all in Baire classes of indices less than ξ.

The investigation of the Borel, projective and Baire hierarchies was initiated a century ago and it is still a very active field. This thesis is concerned with several problems related to the classes defined above.

In Chapter 2 we investigate the properties of negligible sets in Polish topological groups. A Haar measure on a Polish group is basically a well-behaved translation invariant measure. A topological group is locally compact if each of its points has an open neighbourhood whose closure is compact. It is an old result of A. Haar and A. Weil that a Haar measure exists and is essentially unique on locally compact Polish groups, but does not exist on non-locally compact Polish groups. Interestingly, the concept of Haar negligible sets can be generalised to the non-locally compact case as well, this was done first by J. P. R. Christensen [11] and later independently by B. Hunt, T. Sauer and J. Yorke [37]. Hence it is very natural to ask which properties of Haar null sets in locally compact groups remain valid in the non-locally compact setting. Answering questions of D. Fremlin [30] and J. Mycielski [53] we prove that in the non-locally compact case certain regularity properties are not true anymore. In particular, we show that in every abelian non-locally compact Polish group there exists a Borel Haar null set that does not have a $\mathbf{\Pi}^0_2$ (in other words G_δ) Haar null hull.

Chapter 3 tackles the problem of characterisation of the linearly ordered sets of Baire class 1 functions on Polish spaces ordered by the pointwise ordering. If X is a Polish space, the set $\mathcal{B}_1(X)$ forms a partially ordered set (poset) with the pointwise ordering $<_p$, that is, $f <_p g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists an x such that $f(x) < g(x)$. The investigation of this poset was initiated by K. Kuratowski [44], who proved that it contains no uncountable strictly monotone transfinite sequences. Solving a problem that was posed by M. Laczkovich [45] in the 1970s we give a full characterisation of the linearly ordered subsets of this poset in terms of a universal linearly ordered set. Namely, there exists concrete, combinatorially definable linearly ordered set such that a linearly ordered set is order isomorphic to a linearly ordered set of Baire class 1 functions if and only if it can be embedded order preservingly into our universal linearly ordered set. Using this result we easily reprove the theorems of Kuratowski, P. Komjáth, M. Elekes and J. Steprāns (see [44],[43],[19],[23]) and we answer all of the known open questions concerning the linearly ordered sets of Baire class 1 functions. The results of Chapter 2 and 3 are joint work with M. Elekes.

A rank function on a set A is a function that assigns an ordinal to every element of A. Ranks play a central role in descriptive set theory. One can think of a rank as a function that measures the complexity of the elements of the set A, namely the larger the rank of an element, the higher its complexity. A. Kechris and A. Louveau [42] built an extensive theory of ranks defined on the first Baire class, which turned out to be a fundamental tool in the investigation of the functions in the first Baire class. Elekes and Laczkovich [22] asked whether these results can be generalised to the higher Baire classes. They pointed out that this could be used in solving infinite systems of functional equations. We answer their question affirmatively, Chapter 4 deals with defining well-behaved ranks (e.g. subadditive, translation invariant) on the ξth Baire class for $\xi \geq 2$. We also show that surprisingly the most natural approach does not work, however one can construct nice ranks using topology refinements. These are joint results with M. Elekes and V. Kiss.

Chapter 5 is devoted to the precise formulation and generalisation of a method discovered by A. W. Miller [52]. Transfinite induction is a basic tool in contemporary mathematics to construct objects with prescribed properties. In general, the resulting set is not definable and it does not have any regularity properties such as measurability or the Baire property. However, Miller suggested a method to inductively construct, under certain assumptions, such sets that are coanalytic. Coanalytic sets are known to be measurable and they have the Baire property. This method is frequently used ([29], [31], [39] etc.), sometimes omitting the proof, sometimes formulating it in the particular case. We precisely formulate a black box condition that can be applied in such situations without understanding the theories behind Miller's argument. Roughly speaking, our theorem states that it is consistent that if given a transfinite induction that picks a real x_α at each step α, the set of possible choices is nice and large enough then the induction can be realised so that it produces a coanalytic set. Using this theorem we reprove Miller's results and show some new applications as well.

1.1 Notation and basic facts

In this section we collect the notions that are probably all well known to the reader familiar with the basics of descriptive set theory. For the sake of readability, the more specific terminology is explained in the "Preliminaries" section of every chapter. We will mostly follow the notations of the monograph [40].

For a set $H \subset X \times Y$ we define its *x-section* as $H_x = \{y \in Y : (x, y) \in H\}$, and similarly if $H \subset X \times Y \times Z$ then $H_{x,y} = \{z \in Z : (x, y, z) \in H\}$, etc. For a function $f : X \times Y \to Z$ the x-section is the function $f_x : Y \to Z$ defined by $f_x(y) = f(x, y)$. We will sometimes also write $f_x = f(x, \cdot)$.

If H is a set $\mathcal{P}(H)$ stands for the power set of H. An ordinal is identified with the set of its predecessors, for example $2 = \{0, 1\}$.

Let $X = (X, \tau)$ be a Polish space, that is, a separable, completely metrisable topological space endowed with the topology τ. We denote a compatible, complete metric for (X, τ) by d. A *Polish group* is a topological group whose topology is Polish. For a set $H \subset X$ we denote the characteristic function, closure and complement of H by χ_H, $\overline{H}$, and H^c, respectively.

As mentioned above, for a Polish space X, $\mathbf{\Pi}^0_\xi(X)$, $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Delta}^1_1(X)$ etc. stand for the collections of subsets of X in the appropriate classes. We will also use the notation

$$\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X),$$

these are the so called *ξth ambiguous Borel classes*. We say that a set H is *ambiguous* if $H \in \mathbf{\Delta}^0_2(X)$. Symbols Γ and Λ will denote one of the above mentioned classes, and $\check{\Lambda} = \{A^c : A \in \Lambda\}$.

$\mathcal{B}_\xi(X)$ denotes the set of real valued Baire class ξ functions defined on the space X. For a real valued function f on X and a real number c, we let $\{f < c\} = \{x \in X : f(x) < c\}$. We use the notations $\{f > c\}$, $\{f \leq c\}$, $\{f \geq c\}$ and $\{f \neq c\}$ analogously. It is well-known that a function is of Baire class ξ iff the inverse image of every open set is in $\mathbf{\Sigma}^0_{\xi+1}$ iff $\{f < c\}$ and $\{f > c\}$ are in $\mathbf{\Sigma}^0_{\xi+1}$ for every $c \in \mathbb{R}$. For a function $f : X \to \mathbb{R}$ the subgraph of f is the set $sgr(f) = \{(x, r) \in X \times \mathbb{R} : r \leq f(x)\}$.

For $A, B \subset X$ let $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. If X is a group with the operation $+$ let $A + B = \{a + b : a \in A, b \in B\}$. Let us denote by $B(x, r)$ and $\bar{B}(x, r)$ the open and closed ball centred at x of radius r.

$\mathcal{K}(X)$ will stand for the set of non-empty compact subsets of X endowed with the Hausdorff metric. It is well known (see [40, Section 4.F]) that if X is Polish then so is $\mathcal{K}(X)$. Moreover, the compactness of X is equivalent to the compactness of $\mathcal{K}(X)$.

A subset $P \subseteq X$ is *perfect* if it is closed and has no isolated points. A non-empty perfect subset of a Polish space with the subspace topology is an uncountable Polish space.

Chapter 2

Hulls of Haar null sets

Throughout this chapter, let G be an abelian Polish group (the group operation will be denoted by $+$ and the neutral element by 0). It is a well-known result of Birkhoff and Kakutani that any metrisable group admits a left invariant metric [7, 1.1.1], which is clearly two-sided invariant for abelian groups. Moreover, it is also well-known that a two-sided invariant metric on a Polish group is complete [7, 1.2.2]. Hence from now on let d be a fixed complete two-sided invariant metric on G. For the ease of notation we will restrict our attention to abelian groups, but we remark that all our results easily generalise to all Polish groups admitting a two-sided invariant metric.

If G is locally compact then there exists a Haar measure on G, that is, a regular invariant Borel measure that is finite for compact sets and positive for non-empty open sets. This measure, which is unique up to a positive multiplicative constant, plays a fundamental role in the study of locally compact groups. Unfortunately, it is known that non-locally compact Polish groups admit no Haar measure. However, the notion of a Haar nullset has a very well-behaved generalisation. The following definition was invented by Christensen [11], and later rediscovered by Hunt, Sauer and Yorke [37]. (Actually, Christensen's definition was what we call generalised Haar null below, but this subtlety will only play a role later.)

Definition 2.0.1. A set $X \subset G$ is called *Haar null* if there exists a Borel set $B \supset X$ and a Borel probability measure μ on G such that $\mu(B + g) = 0$ for every $g \in G$.

Note that the term *shy* is also commonly used for Haar null, and co-Haar null sets are often called *prevalent*.

Christensen showed that the Haar null sets form a σ-ideal, and also that in locally compact groups a set is Haar null iff it is of measure zero with respect to the Haar measure. During the last two decades Christensen's notion has been very useful in studying exceptional sets in diverse areas such as analysis, functional analysis, dynamical systems, geometric measure theory, group theory, and descriptive set theory.

Therefore it is very important to understand the fundamental properties of this σ-ideal, such as the Fubini properties, ccc-ness, and all other similarities and differences between the locally compact and the general case.

One such example is the following very natural question, which was Problem 1 in Mycielski's celebrated paper [53] more than 20 years ago, and was also discussed e.g. in [18], [4] and [57].

Question 2.0.2. [J. Mycielski] *Let G be a Polish group. Can every Haar null set be covered by a G_δ Haar null set?*

It is easy to see using the regularity of Haar measure that the answer is in the affirmative if G is locally compact.

The first main goal of the present chapter is to answer this question.

Theorem 2.0.3. *If G is a non-locally compact abelian Polish group then there exists a (Borel) Haar null set $B \subset G$ that cannot be covered by a G_δ Haar null set.*

Actually, the proof will immediately yield that G_δ can be replaced by any other class of the Borel hierarchy.

Theorem 2.0.4. *If G is a non-locally compact abelian Polish group and $1 \leq \xi < \omega_1$ then there exists a (Borel) Haar null set $B \subset G$ that cannot be covered by a $\mathbf{\Pi}^0_\xi$ Haar null set.*

It was pointed out to us by Sz. Głąb, see e.g. [8, Proposition 5.2] that an easy but very surprising consequence of this theorem is the following. For the definition of the additivity of an ideal see e.g. [5].

Corollary 2.0.5. *If G is a non-locally compact abelian Polish group then the additivity of the σ-ideal of Haar null sets is ω_1.*

In order to be able to formulate the next question we need to introduce a slightly modified notion of Haar nullness. Numerous authors actually use the following weaker definition, in which B is only required to be universally measurable. (A set is called *universally measurable* if it is measurable with respect to every Borel probability measure. Borel measures are identified with their completions.)

Definition 2.0.6. A set $X \subset G$ is called *generalised Haar null* if there exists a universally measurable set $B \supset X$ and a Borel probability measure μ on G such that $\mu(B + g) = 0$ for every $g \in G$.

In almost all applications X is actually Borel, so it does not matter which of the above two definitions we use. Still, it is of substantial theoretical importance to understand the relation between the two definitions. The next question is from Fremlin's problem list [30].

Question 2.0.7. [D. H. Fremlin, Problem GP] *Is every generalised Haar null set Haar null? In other words, can every generalised Haar null set be covered by a Borel Haar null set?*

Dougherty [18, p.86] showed that under the Continuum Hypothesis or Martin's Axiom the answer is in the negative in every non-locally compact Polish group with a two-sided invariant metric. Later Banakh [4] proved the same under slightly different set-theoretical assumptions. Dougherty uses transfinite induction, and Banakh's proof is basically an existence proof using that the so called cofinality (see e.g. [5] for the definition) of the σ-ideal of generalised Haar null sets is greater than the continuum in some models, hence these examples are clearly very far from being Borel.

The second main goal of the chapter is to answer Fremlin's problem in ZFC.

Since Solecki [57] proved that every analytic generalised Haar null set is contained in a Borel Haar null set, the following result is optimal.

Theorem 2.0.8. *Not every generalised Haar null set is Haar null. More precisely, if G is a non-locally compact abelian Polish group then there exists a coanalytic generalised Haar null set $P \subset G$ that cannot be covered by a Borel Haar null set.*

For more results concerning fundamental properties and applications of Haar null sets in non-locally compact groups see e.g. [2], [3], [14], [16], [17], [24], [35], [50], [58], [61].

2.1 Preliminaries

The following facts can all be found in [40]. Let $\mathcal{F}(G)$ denote the family of closed subsets of G equipped with the so called Effros Borel structure. Recall that $\mathcal{K}(G)$ is the family of compact subsets of G equipped with the Hausdorff metric. Then $\mathcal{K}(G)$ is a Borel subset of $\mathcal{F}(G)$ and the inherited Borel structure on $\mathcal{K}(G)$ coincides with the one given by the Hausdorff metric.

Let us denote by $\mathcal{M}(G)$ the set of Borel probability measures on G, where by Borel probability measure we mean the completion of a probability measure defined on the Borel sets. These measures form a Polish space equipped with the weak*-topology. For $\mu \in \mathcal{M}(G)$ we denote by $\mathrm{supp}(\mu)$ the support of μ, i.e. the minimal closed subset F of G such that $\mu(F) = 1$. Let $\mathcal{M}_c(G) = \{\mu \in \mathcal{M}(G) : \mathrm{supp}(\mu) \text{ is compact}\}$.

Remark 2.1.1. In both versions of the definition of Haar null sets (or generalised Haar null sets) certain authors actually require that the measure μ, which we will often refer to as a *witness measure*, has compact support. This is quite important if the underlying group is non-separable. However, in our case this would make no difference, since in a Polish space for every Borel probability measure there exists a compact set with positive measure [40, 17.11], and then restricting the measure to this set and normalising yields a witness with a compact support. Therefore we may suppose throughout the proofs that our witness measures have compact support.

2.2 The proofs

2.2.1 A function with a surprisingly thick graph

Throughout the proofs, let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1 \leq \xi < \omega_1$, or let $\Gamma = \mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$.

The following result will be the starting point of our constructions. For a fixed measure μ statement (2) below describes the following strange phenomenon: There exists a Borel graph of a function in a product space such that every G_δ cover of the graph has a vertical section of positive measure.

Theorem 2.2.1. *Let* $\Gamma = \mathbf{\Delta}^1_1$ *and* $\Lambda = \mathbf{\Pi}^0_\xi$ *for some* $1 \leq \xi < \omega_1$*, or let* $\Gamma = \mathbf{\Pi}^1_1$ *and* $\Lambda = \mathbf{\Delta}^1_1$*. Then there exists a partial function* $f : \mathcal{M}_c(G) \times 2^\omega \to G$ *with* $\operatorname{graph}(f) \in \Gamma$ *satisfying the following properties:* $\forall \mu \in \mathcal{M}_c(G)$

(1) $(\forall x \in 2^\omega)\,[(\mu, x) \in \operatorname{dom}(f) \Rightarrow f(\mu, x) \in \operatorname{supp}(\mu)]$,

(2) $(\forall S \in \Lambda(2^\omega \times G))\,[(\operatorname{graph}(f_\mu) \subset S \Rightarrow (\exists x \in 2^\omega)(\mu(S_x) > 0)]$.

Before the proof we need several technical lemmas.

Lemma 2.2.2. $\mathcal{M}_c(G)$ *is a Borel subset of* $\mathcal{M}(G)$*.*

Proof. The map $\mu \mapsto \operatorname{supp}(\mu)$ between $\mathcal{M}(G)$ and $\mathcal{F}(G)$ is Borel (see [40, 17.38]) and $\mathcal{M}_c(G)$ is the preimage of $\mathcal{K}(G)$ under this map. □

Lemma 2.2.3. *Let* X *be a Polish space and* $C \subset \mathcal{M}_c(G) \times X \times G$ *with* $C \in \Gamma$*. Then* $\{(\mu, x) : \mu(C_{\mu,x}) > 0\} \in \Gamma$*.*

Proof. Let first $\Gamma = \mathbf{\Delta}^1_1$. If Y is a Borel space and $C \subset Y \times G$ is a Borel set then the map $\varphi\colon Y \times \mathcal{M}_c(G) \to [0, 1]$ defined by $\varphi(y, \mu) = \mu(C_y)$ is Borel ([40, 17.25]). Using this for $Y = \mathcal{M}_c(G) \times X$ we obtain that the map $\psi\colon \mathcal{M}_c(G) \times X \to [0, 1]$ given by $\psi(\mu, x) = \varphi((\mu, x), \mu) = \mu(C_{\mu,x})$ is also Borel. Then $\{(\mu, x) : \mu(C_{\mu,x}) > 0\} = \psi^{-1}((0, 1])$, hence Borel.

For $\Gamma = \mathbf{\Pi}^1_1$ this is simply a special case of [40, 36.23]. □

Lemma 2.2.4. *The set* $\{(\mu, g) : g \in \operatorname{supp}(\mu)\} \subset \mathcal{M}_c(G) \times G$ *is Borel.*

Proof. As mentioned above, the map $\mu \mapsto \operatorname{supp}(\mu)$ is Borel between $\mathcal{M}(G)$ and $\mathcal{F}(G)$, hence its restriction to $\mathcal{M}_c(G)$ is also Borel.

Let $E = \{(K, g) : K \in \mathcal{K}(G),\ g \in K\}$, which clearly is a closed subset of $\mathcal{K}(G) \times G$. If we denote by $\Psi : \mathcal{M}_c(G) \times G \to \mathcal{K}(G) \times G$ the Borel map defined by $(\mu, g) \mapsto (\operatorname{supp}(\mu), g)$ then we obtain that $\{(\mu, g) : g \in \operatorname{supp}(\mu)\} = \Psi^{-1}(E)$ is Borel. □

Let us now prove Theorem 2.2.1.

Proof. Let $U \in \Gamma(2^\omega \times 2^\omega \times G)$ be universal for the $\check{\Lambda}$ subsets of $2^\omega \times G$, that is, for every $A \in \check{\Lambda}(2^\omega \times G)$ there exists an $x \in 2^\omega$ such that $U_x = A$ (for the existence of such a set see [40, 22.3, 26.1]). Notice that $\check{\Lambda} \subset \Gamma$. Let

$$U' = \mathcal{M}_c(G) \times U.$$

Define

$$U'' = \{(\mu, x, g) \in \mathcal{M}_c(G) \times 2^\omega \times G : (\mu, x, x, g) \in U' \text{ and } \mu(U'_{\mu,x,x}) > 0\},$$

then $U'' \in \Gamma$ using that the map $(\mu, x, g) \mapsto (\mu, x, x, g)$ is continuous and by Lemma 2.2.3. Let

$$U''' = \{(\mu, x, g) \in U'' : g \in \operatorname{supp}(\mu)\},$$

then $U''' \in \Gamma$ by Lemma 2.2.4. Clearly,

$$U'''_{\mu,x} = \begin{cases} U'_{\mu,x,x} \cap \operatorname{supp}(\mu) & \text{if } \mu(U'_{\mu,x,x}) > 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since for all (μ, x) the section $U'''_{\mu,x}$ is either empty or has positive μ measure, by the 'large section uniformisation theorem' [40, 18.6] and the coanalytic uniformisation theorem [40, 36.14] there exists a partial function f with $\operatorname{graph}(f) \in \Gamma$ such that $\operatorname{dom}(f) = \{(\mu, x) \in \mathcal{M}_c(G) \times 2^\omega : \mu(U'_{\mu,x,x}) > 0\}$ and $\operatorname{graph}(f) \subset U'''$.

We claim that this f has all the required properties.

First, by the definition of U''', clearly $f(\mu, x) \in \operatorname{supp}(\mu)$ holds whenever $(\mu, x) \in \operatorname{dom}(f)$, hence Property (1) of Theorem 2.2.1 holds.

Let us now prove Property (2). Assume towards a contradiction that there exists $\mu \in \mathcal{M}_c(G)$ and $S \in \Lambda(2^\omega \times G)$ such that $\operatorname{graph}(f_\mu) \subset S$ and $\mu(S_x) = 0$ for every $x \in 2^\omega$. Define $B = (2^\omega \times G) \setminus S$. By the universality of U there exists $x \in 2^\omega$ such that $U_x = U'_{\mu,x} = B$. Now, for every $y \in 2^\omega$ the section B_y is of positive (actually full) μ measure, in particular $\mu(U'_{\mu,x,x}) > 0$, and therefore $(\mu, x) \in \operatorname{dom}(f)$ and

$$f(\mu, x) \in U'''_{\mu,x} \subset U''_{\mu,x} = U'_{\mu,x,x} = B_x.$$

However, $f(\mu, x) \in S_x = G \setminus B_x$, a contradiction. □

2.2.2 Translating the compact sets apart

This section heavily builds on ideas of Solecki [56], [57]. The main point is that if G is non-locally compact then one can apply a translation (chosen in a Borel way) to every compact subset of G so that the resulting translates are disjoint. (For technical reasons we will need to consider continuum many copies of each compact set and also to "blow them up" by a fixed compact set C.)

Proposition 2.2.5. *Let $C \in \mathcal{K}(G)$ be fixed. Then there exists a Borel map $t : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to G$ such that*

(1) if $(K, x, y) \neq (K', x', y')$ are elements of $\mathcal{K}(G) \times 2^\omega \times 2^\omega$ then

$$(K - C + t(K, x, y)) \cap (K' - C + t(K', x', y')) = \emptyset$$

(2) for every $K \in \mathcal{K}(G)$ and $y \in 2^\omega$ the map $t(K, \cdot, y)$ is continuous.

Proof. We use Solecki's arguments [56], [57], which he used for different purposes, with some modifications. However, for the sake of completeness, we repeat large parts of his proofs.

Fix an increasing sequence of finite sets $Q_k \subset G$ with $0 \in Q_0$ such that $\cup_{k\in\omega} Q_k$ is dense in G.

Lemma 2.2.6. *For every $\varepsilon > 0$ there exists $\delta > 0$ and a sequence $\{g_k\}_{k\in\omega} \subset B(0, \varepsilon)$ such that for every distinct $k, k' \in \omega$*

$$d(Q_k + g_k, Q_{k'} + g_{k'}) \geq \delta.$$

Proof. Since G is not locally compact, there exists $\delta > 0$ and a countably infinite set $S \subset B(0, \varepsilon)$ such $d(s, s') \geq 2\delta$ for every distinct $s, s' \in S$.

Now we define g_k inductively as follows. Suppose that we are done for $i < k$. If for every $s \in S$ there are $a \in Q_k$, $i < k$ and $b \in Q_i$ with $d(a + s, b + g_i) < \delta$ then there is a pair s, s' of distinct members of S with the same a, i and b. But then

$$d(s, s') = d(a + s, a + s') \leq d(a + s, b + g_i) + d(b + g_i, a + s') < 2\delta,$$

a contradiction. Hence we can let $g_k = s$ for an appropriate $s \in S$. □

It is easy to see that using the previous lemma repeatedly we can inductively fix ε_n, $\delta_n < \varepsilon_n$ and sequences $\{g^n_k\}_{k\in\omega}$ such that for every $n \in \omega$

- $\{g^n_k\}_{k\in\omega} \subset B(0, \varepsilon_n)$,
- $d(Q_k + g^n_k, Q_{k'} + g^n_{k'}) \geq 2\delta_n$ for every distinct $k, k' \in \omega$,
- $\sum_{m>n} \varepsilon_m < \frac{\delta_n}{3}$.

Note that the second property implies that for every $n \in \omega$ the function $k \mapsto g^n_k$ is injective. Note also that $\varepsilon_n \to 0$ and hence $\delta_n \to 0$, moreover, $\sum \delta_n$ is also convergent.

Let us also fix a Borel injection $c : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to \omega^\omega$ such that for each K and y the map $c(K, \cdot, y)$ is continuous. (E.g. fix a Borel injection $c_1 : \mathcal{K}(G) \to 2^\omega$ and continuous injection $c_2 : 2^\omega \times 2^\omega \times 2^\omega \to \omega^\omega$ and let $c(K, x, y) = c_2(c_1(K), x, y)$.)

Our goal now is to define $t(K, x, y)$, so let us fix a triple (K, x, y). First we define a sequence $\{h_n = h_n(K, x, y)\}_{n\in\omega}$ with $h_n \in \{g^n_k\}_{k\in\omega}$ as follows. Suppose that we are given h_i for $i < n$. By the density of $\cup_k Q_k$ we have $G = \cup_k (Q_k + B(0, \delta_n/2))$. Since $K - C$ is compact, there exists a minimal index $k_n(K, x, y)$ such that

$$K - C + \sum_{i<n} h_i \subset Q_{k_n(K,x,y)} + B(0, \delta_n/2).$$

Fix an injective map $\phi : \omega \times \omega \to \omega$ with $\phi(i, j) \geq i$ for every $i \in \omega$ and let

$$h_n = g^n_{\phi(k_n(K,x,y),c(K,x,y)(n))} \tag{2.2.1}$$

and

$$t(K, x, y) = \sum_{n\in\omega} h_n. \tag{2.2.2}$$

We claim that this function has the required properties.

First, it is well defined, that is, the sum is convergent since $h_n \in B(0, \varepsilon_n)$, and hence for all $n \in \omega$

$$\sum_{m>n} h_m \in \bar{B}(0, \delta_n/3). \tag{2.2.3}$$

In order to prove (1) of the Proposition, let us now fix $(K, x, y) \neq (K', x', y')$. Then there exists an $n \in \omega$ such that $c(K, x, y)(n) \neq c(K', x', y')(n)$. By the injectivity of ϕ and of the sequence $k \mapsto g^n_k$ and also by (2.2.1) we obtain that $h_n(K, x, y) \neq h_n(K', x', y')$. Denote by h_i and h'_i the elements $h_i(K, x, y)$ and $h_i(K', x', y')$, respectively. Set

$$k = \phi(k_n(K, x, y), c(K, x, y)(n)) \text{ and } k' = \phi(k_n(K', x', y'), c(K', x', y')(n)).$$

The condition $\phi(i, j) \geq i$ implies $k \geq k_n(K, x, y)$, hence $Q_k \supset Q_{k_n(K,x,y)}$ and similarly $k' \geq k_n(K', x', y')$, so $Q_{k'} \supset Q_{k_n(K',x',y')}$. Therefore, by the definition of k_n,

$$K - C + \sum_{i<n} h_i \in Q_k + B(0, \delta_n/2) \text{ and } K' - C + \sum_{i<n} h'_i \in Q_{k'} + B(0, \delta_n/2),$$

hence

$$K - C + \sum_{i\leq n} h_i \in Q_k + h_n + B(0, \delta_n/2) \text{ and } K' - C + \sum_{i\leq n} h'_i \in Q_{k'} + h'_n + B(0, \delta_n/2).$$

Thus, using the triangle inequality and the second property of the g^n_k we obtain

$$d(K - C + \sum_{i\leq n} h_i, K' - C + \sum_{i\leq n} h'_i) \geq d(Q_k + h_n, Q_{k'} + h'_n) - 2 \cdot \frac{\delta_n}{2} =$$

$$= d(Q_k + g^n_k, Q_{k'} + g^n_{k'}) - \delta_n \geq 2\delta_n - \delta_n = \delta_n.$$

From this, using (2.2.3), we obtain $d(K-C+t(K, x, y), K'-C+t(K', x', y')) \geq \delta_n - 2\frac{\delta_n}{3} = \frac{\delta_n}{3} > 0$, which proves (1).

What remains to show is that t is a Borel map and for every K and y the map $t(K, \cdot, y)$ is continuous. But (2.2.3) shows that the series defining t in (2.2.2) is uniformly convergent, so the next lemma finishes the proof.

Lemma 2.2.7. *For every $n \in \omega$ the map h_n is Borel and for every K and y the map $h_n(K, \cdot, y)$ is continuous.*

Proof. We will actually prove more by induction on n. Define $f_n \colon \mathcal{K}(G) \times 2^\omega \times 2^\omega \to \mathcal{K}(G)$ by

$$f_n(K, x, y) = K - C + \sum_{i<n} h_i(K, x, y). \tag{2.2.4}$$

We claim that the maps f_n, k_n and h_n are Borel and for every K and y the maps $f_n(K, \cdot, y)$, $k_n(K, \cdot, y)$ and $h_n(K, \cdot, y)$ are locally constant.

Note that if a function takes its values from a discrete set than locally constant is equivalent to continuous.

First we prove that the maps are Borel. Suppose that we are done for $i < n$. Let us check that f_n is Borel. Put $\eta : (K, x, y) \mapsto (K, \sum_{i<n} h_i(K, x, y))$ and $\psi : (K, g) \mapsto K - C + g$, then $f_n = \psi \circ \eta$. Moreover, η is Borel by induction, and ψ is easily seen to be continuous, hence f_n is Borel.

Next we show that k_n is Borel. Since $\operatorname{ran}(k_n) \subset \omega$, we need to check that for every fixed $m \in \omega$ the set $B = \{(K, x, y) \colon k_n(K, x, y) = m\}$ is Borel. By the definition of $k_n(K, x, y)$, clearly

$$B = \{(K, x, y) \colon f_n(K, x, y) \subset U \text{ and } f_n(K, x, y) \not\subset V\},$$

where $U = Q_m + B(0, \delta_n/2)$ and $V = Q_{m-1} + B(0, \delta_n/2)$ are fixed open sets.

Set $\mathcal{U}_W = \{L \in \mathcal{K}(G) : L \subset W\}$, which is open in $\mathcal{K}(G)$ for every open set $W \subset G$. Then clearly

$$B = f_n^{-1}(\mathcal{U}_U) \setminus f_n^{-1}(\mathcal{U}_V),$$

hence Borel.

Since the functions $k \mapsto g_k^n$ and ϕ defined on countable sets are clearly Borel, the Borelness of k_n and c imply by (2.2.1) that h_n is also Borel.

In order to prove that f_n, k_n and h_n are locally constant in the second variable, fix K and y and suppose that we are done for $i < n$. Then (2.2.4) shows that f_n is locally constant in the second variable by induction. This easily implies using the definition of k_n that k_n is also locally constant in the second variable. But from this, and from the fact that $c(K, \cdot, y)(n) \colon 2^\omega \to \omega$ is continuous, hence locally constant, it is also clear using (2.2.1) that h_n is also locally constant in the second variable, which finishes the proof of the Lemma. □

Therefore the proof of the Proposition is also complete. □

2.3 Putting the ingredients together

Now we are ready to prove the main results of this chapter, which are summarised in the following theorem.

Theorem 2.3.1. *Let* $\Gamma = \mathbf{\Delta}^1_1$ *and* $\Lambda = \mathbf{\Pi}^0_\xi$ *for some* $1 \leq \xi < \omega_1$*, or let* $\Gamma = \mathbf{\Pi}^1_1$ *and* $\Lambda = \mathbf{\Delta}^1_1$*. If* G *is a non-locally compact abelian Polish group then there exists a (generalised, in the case of* $\Gamma = \mathbf{\Pi}^1_1$*) Haar null set* $E \in \Gamma(G)$ *that is not contained in any Haar null set* $H \in \Lambda(G)$*.*

Proof. Let f be given by Theorem 2.2.1.

Denote the Borel map $\mu \mapsto \mathrm{supp}(\mu)$ by $\mathrm{supp} : \mathcal{M}_c(G) \to \mathcal{K}(G)$. Let us also fix a Borel bijection $c : \mathcal{M}_c(G) \to 2^\omega$ (which we think of as a coding map) and a continuous probability measure ν on G with compact support C containing 0 (compactly supported continuous measures exist on every Polish space without isolated points, since such spaces contain copies of 2^ω). Let $t : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to G$ be the map from Proposition 2.2.5 with the C fixed above, and define the map $\Psi \colon \mathcal{M}_c(G) \times 2^\omega \times G \to G$ by

$$\Psi(\mu, x, g) = g + t(\mathrm{supp}(\mu), x, c(\mu)). \tag{2.3.1}$$

Finally, define $E = \Psi(\mathrm{graph}(f))$.

Claim 2.3.2. $E \in \Gamma$*.*

Proof. Ψ is clearly a Borel map. We claim that it is injective on $D = \{(\mu, x, g) : \mu \in \mathcal{M}_c(G), g \in \mathrm{supp}(\mu)\}$, which is Borel by Lemma 2.2.2 and Lemma 2.2.4. Let $(\mu, x, g) \neq (\mu', x', g')$ be elements of D, we need to check that Ψ takes distinct values on them. The case $(\mu, x) = (\mu', x')$ is obvious, while the case $(\mu, x) \neq (\mu', x')$ follows from Property (1) in Proposition 2.2.5, since $\Psi(\mu, x, g) \in \mathrm{supp}(\mu) - C + t(\mathrm{supp}(\mu), x, c(\mu))$ (recall that $g \in \mathrm{supp}(\mu)$ and $0 \in C$). Therefore Ψ is a Borel isomorphism on D. By $\mathrm{graph}(f) \subset D$ this implies that $E = \Psi(\mathrm{graph}(f))$ is in Γ (for $\Gamma = \mathbf{\Delta^1_1}$ see [40, 15.4], for $\Gamma = \mathbf{\Pi}^1_1$ notice that by [40, 25.A] a Borel isomorphism takes analytic sets to analytic sets, hence coanalytic sets to coanalytic sets). □

Claim 2.3.3. E *is Haar null (generalised Haar null in the case of* $\Gamma = \mathbf{\Pi}^1_1$*).*

Proof. We prove that ν is witnessing this fact. Actually, we prove more: $|C \cap (E+g)| \leq 1$ for every $g \in G$, or equivalently $|(C + g) \cap E| \leq 1$ for every $g \in G$. So let us fix $g \in G$.

$$E = \Psi(\mathrm{graph}(f)) = \{\Psi(\mu, x, f(\mu, x)) : (\mu, x) \in \mathrm{dom}(f)\} =$$

$$\{f(\mu, x) + t(\mathrm{supp}(\mu), x, c(\mu)) : (\mu, x) \in \mathrm{dom}(f)\},$$

hence the elements of E are of the form $g^{\mu,x} = f(\mu, x) + t(\mathrm{supp}(\mu), x, c(\mu))$. This element $g^{\mu,x}$ is clearly in $A^{\mu,x} = \mathrm{supp}(\mu) + t(\mathrm{supp}(\mu), x, c(\mu))$ by Property (1) of Theorem 2.2.1, and the sets $A^{\mu,x}$ form a pairwise disjoint family as (μ, x) ranges over $\mathrm{dom}(f)$, by Property (1) of Proposition 2.2.5. Hence it suffices to show that $C + g$ can intersect at most one $A^{\mu,x}$. But it can actually intersect at most one set of the form $K + t(K, x, y)$, since otherwise g would be in the intersection of two distinct sets of the form $K - C + t(K, x, y)$, contradicting Property (1) of Proposition 2.2.5. □

Claim 2.3.4. *There is no Haar null set* $H \in \Lambda$ *containing* E*.*

Suppose that $H \in \Lambda$ is such a set. Then by Remark 2.1.1 there exists a probability measure μ with compact support witnessing this fact. The section map $\Psi_\mu = \Psi(\mu, \cdot, \cdot)$ is continuous by (2.3.1) and Property (2) of Proposition 2.2.5. Now let $S = \Psi_\mu^{-1}(H)$, then $S \in \Lambda(2^\omega \times G)$.

It is easy to check that $\text{graph}(f_\mu) \subset S$, and therefore, using Theorem 2.2.1, there exists $x \in 2^\omega$ such that $\mu(S_x) > 0$. By the definition of S we have that $\Psi(\mu, x, S_x) \subset \Psi_\mu(S) \subset H$. But $\Psi(\mu, x, \cdot) : G \to G$ is a translation, so a translate of H contains S_x, which is of positive μ measure, contradicting that H is Haar null with witness μ. □

This concludes the proof.

2.4 Open problems

Question 2.4.1. *Let G be a non-locally compact abelian Polish group. Does there exist an F_σ Haar null set that cannot be covered by a G_δ Haar null set?*

Interestingly, our proof does not give any information about the Borel class of our example.

Question 2.4.2. *What is the least complexity of such a set? And in general, what is the least complexity of a Haar null set that cannot be covered by a $\mathbf{\Pi}^0_\xi$ Haar null set?*

Remark 2.4.3. We remark that it is not hard to show that in abelian Polish groups every σ-compact Haar null set can be covered by a G_δ Haar null set.

Question 2.4.4. *Do the results of the chapter hold in all (not necessarily abelian) non-locally compact Polish groups?*

Question 2.4.5. *Does there exist a Polish group with a countable subset that cannot be covered by a G_δ Haar null set?*

In view of the above remark, the group in the last question cannot be abelian. Of course, it also cannot be locally compact. How about e.g. an arbitrary countable dense subset of $Homeo[0, 1]$? This is actually closely related to the following question, popularised by U. B. Darji, and considered e.g. in [12].

Question 2.4.6. *Can every uncountable Polish group be written as a union of a Haar null set and a meagre set?*

The answer is affirmative e.g. for abelian groups or for groups with a two-sided invariant metric.

The so called cardinal invariants convey a lot of information about the set-theoretical properties of a σ-ideal, see e.g. [5]. Banakh examined this problem in detail in [4] for the σ-ideal of *generalised* Haar null sets.

Question 2.4.7. *What can we say about the cardinal invariants of the σ-ideal of Haar null sets? How about e.g. if $G = \mathbb{Z}^\omega$?*

Surprisingly, the invariants may differ for Haar null and generalised Haar null sets. First, in contrast with Corollary 2.0.5, [4, Thm. 3] shows that the additivity of the generalised Haar null sets in $\mathbb{Z}^\omega$ equals the additivity of the Lebesgue null sets. Second, [4, Thm. 3] also shows that the cofinality of the generalised Haar null sets in $\mathbb{Z}^\omega$ may exceed the continuum, whereas for Haar null sets it is clearly at most continuum.

In separable Banach spaces there is a well-known alternative notion of nullness. For the equivalent definitions of Aronszajn null, cube null and Gaussian null sets see [13].

Question 2.4.8. *Suppose that G is a separable Banach space. Which results of the chapter remain valid when Haar null is replaced by Aronszajn null?*

Chapter 3

Order types representable by Baire class 1 functions

Let $\mathcal{F}(X)$ be a class of real valued functions defined on a Polish space X, e.g. $C(X)$, the set of continuous functions. The natural partial ordering on this space is the pointwise ordering $<_p$, that is, we say that $f <_p g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists at least one x such that $f(x) < g(x)$. If we would like to understand the structure of this partially ordered set (poset), the first step is to describe its linearly ordered subsets.

For example, if $X = [0,1]$ and $\mathcal{F}(X) = \mathcal{C}([0,1])$ then it is a well known result that the possible order types of the linearly ordered subsets of $\mathcal{C}([0,1])$ are the real order types (that is, the order types of the subsets of the reals). Indeed, a real order type is clearly representable by constant functions, and if $\mathcal{L} \subset \mathcal{C}([0,1])$ is a linearly ordered family of continuous functions then (by continuity) $f \mapsto \int_0^1 f$ is a *strictly* monotone map of $\mathcal{L}$ into the reals.

The next natural class to look at is the class of Lebesgue measurable functions. However, it is not hard to check that the assumption of measurability is rather meaningless here. Indeed, if $\mathcal{L}$ is a linearly ordered family of *arbitrary* real functions and $\varphi : \mathbb{R} \to \mathbb{R}$ is a map that maps the Cantor set onto $\mathbb{R}$ and is zero outside of the Cantor set then $f \mapsto f \circ \varphi$ is a strictly monotone map of $\mathcal{L}$ into the class of Lebesgue measurable functions.

Therefore it is more natural to consider the class of Borel measurable functions. However, P. Komjáth [43] proved that it is already independent of ZFC (the usual axioms of set theory) whether the class of Borel measurable functions contains a strictly increasing transfinite sequence of length ω_2.

The next step is therefore to look at subclasses of the Borel measurable functions, namely the Baire hierarchy. Komjáth actually also proved that in his above mentioned result the set of Borel measurable function can be replaced by the set of Baire class 2 functions. This explains why the Baire class 1 case seem to be the most interesting one.

Back in the 1970s M. Laczkovich [45] posed the following problem:

Problem 3.0.1. *Characterise the order types of the linearly ordered subsets of* $(\mathcal{B}_1(X), <_p)$.

We will use the following notation:

Definition 3.0.2. Let $(P, <_P)$ and $(Q, <_Q)$ be two posets. We say that P is *embeddable into* Q, in symbols $(P, <_P) \hookrightarrow (Q, <_Q)$ if there exists a map $\Phi : P \to Q$ so that for every $p, q \in P$ if $p <_P q$ then $\Phi(p) <_Q \Phi(q)$. (Note that an embedding may not be 1-to-1 in general. However, an embedding of a *linearly* ordered set *is* 1-to-1.) If $(L, <_L)$ is a linear ordering and $(L, <_L) \hookrightarrow (Q, <_Q)$ then we also say that L *is representable in* Q.

Whenever the ordering of a poset $(P, <_P)$ is clear from the context we will use the notation $P = (P, <_P)$. Moreover, when Q is not specified, the term "representable" will refer to representability in $\mathcal{B}_1(X)$.

The earliest result that is relevant to Laczkovich's problem is due to Kuratowski. He showed that for any Polish space X we have $\omega_1, \omega_1^* \not\hookrightarrow \mathcal{B}_1(X)$, or in other words, there is no ω_1-long strictly increasing or decreasing sequence of Baire class 1 functions (see [44, §24. III.2.]).

It seems conceivable at first sight that this is the only obstruction, that is, every linearly ordered set that does not contain ω_1-long strictly increasing or decreasing sequences is representable in $\mathcal{B}_1(\mathbb{R})$. First, answering a question of Gerlits and Petruska, this conjecture was consistently refuted by P. Komjáth [43] who showed that no Suslin line (ccc linearly ordered set that is not separable) is representable in $\mathcal{B}_1(\mathbb{R})$. Komjáth's short and elegant proof uses the very difficult set-theoretical technique of forcing. Laczkovich [46] asked if a forcing-free proof exists.

Elekes and Steprāns [23] continued this line of research. On the one hand they proved that consistently Kuratowski's result is a characterisation for order types of cardinality $< \mathfrak{c}$. On the other hand they strengthened Komjáth's result by constructing in ZFC a linearly ordered set L not containing Suslin lines or ω_1-long strictly increasing or decreasing sequences such that L is not representable in $\mathcal{B}_1(X)$.

Among other results, M. Elekes [19] proved that if X and Y are both uncountable σ-compact or both not σ-compact Polish spaces then for a linearly ordered set L we have $L \hookrightarrow \mathcal{B}_1(X) \iff L \hookrightarrow \mathcal{B}_1(Y)$. He asked whether this still holds if X is an uncountable σ-compact Polish space but Y is not σ-compact. Elekes also asked whether the same linearly ordered sets can be embedded into the set of *characteristic* functions in $\mathcal{B}_1(X)$ as into $\mathcal{B}_1(X)$. Notice that a characteristic function χ_A is of Baire class 1 if and only if $A \in \mathbf{\Delta}^0_2(X)$. Moreover, $\chi_A <_p \chi_B \iff A \subsetneq B$, hence the above question is equivalent to whether $L \hookrightarrow (\mathcal{B}_1(X), <_p)$ implies $L \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$. He also asked if *duplications* and completions of representable orders are themselves representable, where the duplication of L is $L \times \{0, 1\}$ ordered lexicographically.

Our main aim in this chapter is to solve Problem 3.0.1 and consequently answer the above mentioned questions. The solution proceeds by constructing a *universal* linearly ordered set for $\mathcal{B}_1(X)$, that is, a linear order that is representable in $\mathcal{B}_1(X)$ such that every representable linearly ordered set is embeddable into it. Of course such a linear order only provides a useful characterisation if it is sufficiently simple combinatorially to work with. We demonstrate this by providing new, simpler proofs of the known theorems (including a forcing-free proof of Komjáth's theorem), and also by answering the above mentioned open questions.

The universal linear ordering can be defined as follows.

Definition 3.0.3. Let $[0,1]^{<\omega_1}_{\searrow 0}$ be the set of strictly decreasing well-ordered transfinite sequences in $[0,1]$ with last element zero. Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{\searrow 0}$ be distinct and let δ be the minimal ordinal such that $x_\delta \neq x'_\delta$. We say that

$$(x_\alpha)_{\alpha \leq \xi} <_{altlex} (x'_\alpha)_{\alpha \leq \xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Now we can formulate the main result of this chapter.

Theorem 3.0.4. *(Main Theorem) Let X be an uncountable Polish space. Then the following are equivalent for a linear ordering $(L,<)$:*

(1) $(L,<) \hookrightarrow (\mathcal{B}_1(X), <_p)$,

(2) $(L,<) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$.

In fact, $(\mathcal{B}_1(X), <_p)$ and $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$ are embeddable into each other.

Using this theorem one can reduce every question concerning the linearly ordered subsets of $\mathcal{B}_1(X)$ to a purely combinatorial problem. We were able to answer all of the known such questions and we reproved easily the known theorems as well. The most important results are:

- Answering a question of Laczkovich [46], we give a new, forcing free proof of Komjáth's theorem. (Theorem 3.3.2)
- The class of ordered sets representable in $\mathcal{B}_1(X)$ does not depend on the uncountable Polish space X. (Corollary 3.2.15)
- There exists an embedding $(\mathcal{B}_1(X), <_p) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$, hence a linear ordering is representable by Baire class 1 functions iff it is representable by Baire class 1 *characteristic* functions. (Corollary 3.2.14)
- The duplication of a representable linearly ordered set is representable. More generally, countable lexicographical products of representable sets are representable. (Corollary 3.4.5 and Theorem 3.4.2)
- There exists a linearly ordered set that is representable in $\mathcal{B}_1(X)$ but none of its completions are representable. (Theorem 3.4.12)

The chapter is organised as follows. In Section 3.2 we first prove that there exists an embedding $\mathcal{B}_1(X) \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$, then that $[0,1]^{<\omega_1}_{\searrow 0} \hookrightarrow \mathcal{B}_1(X)$. The former result heavily builds on a theorem of Kechris and Louveau. Unfortunately for us, they only consider the case of compact Polish spaces, while it is of crucial importance in our proof to use their theorem for arbitrary Polish spaces. Moreover, their proof seems to contain a slight error. Hence it was unavoidable to reprove their result, which is the content of Section 3.5. Section 3.3 contains the new proofs of the known results, while in Section 3.4 we answer the above open questions. Finally, in Section 3.6 we formulate some new open problems.

3.1 Preliminaries

Our terminology in this chapter will mostly follow [40] and [59].

Let X be a Polish space. $USC(X)$ stands for the set of *upper semicontinuous* functions, that is, the set of functions f for which for every $r \in \mathbb{R}$ the set $f^{-1}((-\infty, r))$ is open in X. It is easy to see that the infimum of USC functions is also USC.

If $\mathcal{F}(X)$ is a class of real valued functions then we will denote by $b\mathcal{F}(X)$ and $\mathcal{F}^+(X)$ the set of bounded and nonnegative functions in $\mathcal{F}(X)$, respectively.

Recall the following equivalent definition of the first Baire class: $f \in \mathcal{B}_1(X) \iff$ the preimage of every open set under f is in $\mathbf{\Sigma}^0_2(X)$. This easily implies that a characteristic function χ_A is of Baire class 1 if and only if $A \in \mathbf{\Delta}^0_2(X)$. The above equivalent definition also implies that USC functions are of Baire class 1.

Let $(P, <_p)$ be a poset. Let us introduce the following notation for the set of well-ordered sequences in P:

$$\sigma P = \{F : \alpha \to P \mid \alpha \text{ is an ordinal, } F \text{ is strictly increasing}\}.$$

We will use the notation $\sigma^* P$ for the reverse well-ordered sequences, that is,

$$\sigma^* P = \{F : \alpha \to P \mid \alpha \text{ is an ordinal, } F \text{ is strictly decreasing}\}.$$

Then $\sigma^*[0,1]$ is the set of strictly decreasing well-ordered transfinite sequences of reals in $[0,1]$.

For a poset P, if $\bar{p} \in \sigma^* P$ and the domain of $\bar{p}$ is ξ then we will write $\bar{p}$ as $(p_\alpha)_{\alpha<\xi}$, where $p_\alpha = \bar{p}(\alpha)$. We will call the ordinal ξ the length of $\bar{p}$, in symbols $l(\bar{p})$.

Let H and H' be two subsets of the linearly ordered set $(L, <_L)$. We will say that $H \leq_L H'$ or $H <_L H'$ if for every $h \in H$ and $h' \in H'$ we have $h \leq_L h'$ or $h <_L h'$, respectively.

Now if $\bar{p}, \bar{p}' \in \sigma^* P$ and $\bar{p} \not\subset \bar{p}', \bar{p}' \not\subset \bar{p}$ then there exists a minimal ordinal δ so that $p_\delta \neq p'_\delta$. This ordinal is denoted by $\delta(\bar{p}, \bar{p}')$.

Le α be a successor ordinal, then $\alpha - 1$ will stand for its predecessor. Now, since every ordinal α can be uniquely written in the form $\alpha = \gamma + n$ where γ is limit and n is finite, we let $(-1)^\alpha = (-1)^n$ and refer to the parity of n as the *parity of* α.

A poset $(T, <_T)$ is called a *tree* if for every $t \in T$ the ordering $<_T$ restricted to the set $\{s : s <_T t\}$ is a well-ordering. We denote by $Lev_\alpha(T)$ the *αth level of* T, that is, the set $\{t \in T : <_T |_{\{s: s<_T t\}}$ has order type $\alpha\}$. An *α-chain* C is a subset of a tree such that $<_T |_C$ is a well-ordering in type α, whereas an *antichain* is a set that consists of $<_T$-incomparable elements. A set $D \subset T$ is called *dense* if for every $t \in T$ there exists a $p \in D$ such that $t \leq_T p$. A set is called *open* if if for every $p \in D$ we have $\{t \in T : t \geq_T p\} \subset D$.

A tree $(T, <_T)$ of cardinality $\aleph_1$ is called an *Aronszajn tree*, if for every $\alpha < \omega_1$ we have $|Lev_\alpha(T)| \leq \aleph_0$ and T contains no ω_1-chains. An Aronszajn tree is called a *Suslin tree* if it contains no uncountable antichains.

A *Suslin line* is a linearly ordered set that is ccc (it contains no uncountable pairwise disjoint collection of non-empty open intervals) but not separable.

We will call a poset $(P, <_P)$ $\mathbb{R}$-*special* ($\mathbb{Q}$-*special*) if there exists an embedding $P \hookrightarrow \mathbb{R}$ ($P \hookrightarrow \mathbb{Q}$).

3.2 The main result

3.2.1 $\mathcal{B}_1(X) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$

Recall that

$$[0,1]^{<\omega_1}_{\searrow 0} = \{\bar{x} \in \sigma^*[0,1] : \min \bar{x} = 0\}$$

and also that for $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0,1]^{<\omega_1}_{\searrow 0}$ distinct and $\delta = \delta(\bar{x}, \bar{x}')$ we say that

$$(x_\alpha)_{\alpha \leq \xi} <_{altlex} (x'_\alpha)_{\alpha \leq \xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Theorem 3.2.1. *Let X be a Polish space. Then $\mathcal{B}_1(X) \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$.*

In order to prove the theorem we have to make some preparation. We will use results of Kechris and Louveau [42]. They developed a method to decompose a Baire class 1 function into a sum of a transfinite alternating series, which is analogous to the well known Hausdorff-Kuratowski analysis of $\mathbf{\Delta}^0_2$ sets.

First we define the generalised sums.

Definition 3.2.2. ([42]) Suppose that $(f_\beta)_{\beta<\alpha}$ is a pointwise decreasing sequence of nonnegative bounded USC functions for an ordinal $\alpha < \omega_1$. Let us define the *generalised alternating sum* $\sum^*_{\beta<\alpha}(-1)^\beta f_\beta$ by induction on α as follows:

$$\textstyle\sum^*_{\beta<0}(-1)^\beta f_\beta = 0$$

and

$$\textstyle\sum^*_{\beta<\alpha}(-1)^\beta f_\beta = \sum^*_{\beta<\alpha-1}(-1)^\beta f_\beta + (-1)^{\alpha-1} f_{\alpha-1}$$

if α is a successor and

$$\textstyle\sum^*_{\beta<\alpha}(-1)^\beta f_\beta = \sup\{\sum^*_{\gamma<\beta}(-1)^\gamma f_\gamma : \beta < \alpha, \beta \text{ even}\}$$

if $\alpha > 0$ is a limit.

Every nonnegative bounded Baire class 1 function can be canonically decomposed into such a sum. For this we need the notion of upper regularisation.

Definition 3.2.3. ([42]) Let $f : X \to \mathbb{R}$ be a nonnegative bounded function. The *upper regularisation* of f is defined as

$$\hat{f} = \inf\{g : f \leq_p g, g \in \mathrm{USC}(X)\}.$$

Note that $\hat{f}$ is USC, since the infimum of USC functions is USC. Also, clearly $\hat{f} = f$ if f is USC.

Definition 3.2.4. ([42])

Let

$$g_0 = f, f_0 = \widehat{g_0},$$

if α is a successor then let

$$g_\alpha = f_{\alpha-1} - g_{\alpha-1}, f_\alpha = \widehat{g}_\alpha,$$

if $\alpha > 0$ is a limit then let

$$g_\alpha = \inf_{\substack{\beta<\alpha \\ \beta \text{ even}}} g_\beta \text{ and } f_\alpha = \widehat{g}_\alpha.$$

Now if there exists a minimal ξ such that $f_\xi \equiv f_{\xi+1}$ then let $\Phi(f) = (f_\alpha)_{\alpha\leq\xi}$.

Note that we need some results of Kechris and Louveau for arbitrary Polish spaces, however in [42] the authors proved the theorems only in the compact Polish case, although the proofs still work for the general case as well. Unfortunately, in our proof the non-σ-compact statement plays a significant role, hence we must check the validity of their results on such spaces. The results used are summarised in Proposition 3.2.5 and the proof can be found in Section 3.5. Notice that the original proof seems to contain a small error, but it can be corrected with the same ideas.

Proposition 3.2.5. *([42]) Let X be a Polish space and $f \in b\mathcal{B}_1^+(X)$. Then $\Phi(f)$ is defined, $\Phi(f) \in \sigma^* bUSC^+$ and we have*

(1) $f = \sum^*_{\beta<\alpha}(-1)^\beta f_\beta + (-1)^\alpha g_\alpha$ *for every* $\alpha \leq \xi$,

(2) $f_\xi \equiv 0$,

(3) $f = \sum^*_{\alpha<\xi}(-1)^\alpha f_\alpha$.

Proof. See Section 3.5. □

Proposition 3.2.6. *Let X be a Polish space and $f_0, f_1 \in b\mathcal{B}_1^+(X)$. Suppose that $f_0 <_p f_1$ and let $\Phi(f_0) = (f^0_\alpha)_{\alpha \leq \xi_0}$ and $\Phi(f_1) = (f^1_\alpha)_{\alpha \leq \xi_1}$. Then $\Phi(f_0) \neq \Phi(f_1)$ and if $\delta = \delta(\Phi(f_0), \Phi(f_1))$ then $f^0_\delta <_p f^1_\delta$ if δ is even and $f^0_\delta >_p f^1_\delta$ if δ is odd.*

Proof. First notice that if $f_0 \neq f_1$ then by (3) of Proposition 3.2.5 we have $\Phi(f_0) \neq \Phi(f_1)$. Let $(g^0_\beta)_{\beta \leq \xi_0}$ and $(g^1_\beta)_{\beta \leq \xi_1}$ be the appropriate sequences (used in Definition 3.2.4 with $\widehat{g^i_\beta} = f^i_\beta$).

We show by induction on β that for every even ordinal $\beta \leq \delta$ we have $g^0_\beta \leq_p g^1_\beta$ and for every odd ordinal $\beta \leq \delta$ we have $g^0_\beta \geq_p g^1_\beta$.

For $\beta = 0$ by definition $g^0_0 = f_0$ and $g^1_0 = f_1$, so $g^0_0 \leq_p g^1_0$.

Suppose that we are done for every $\gamma < \beta$.

- For limit β we have that
$$g^0_\beta = \inf_{\substack{\gamma<\beta \\ \gamma \text{ even}}} g^0_\gamma$$
so by the inductive hypothesis obviously $g^0_\beta \leq_p g^1_\beta$.
- If β is an odd ordinal, since $\beta - 1 < \delta$ we have $f^0_{\beta-1} = f^1_{\beta-1}$ so
$$g^0_\beta = f^0_{\beta-1} - g^0_{\beta-1} \geq_p f^0_{\beta-1} - g^1_{\beta-1} = f^1_{\beta-1} - g^1_{\beta-1} = g^1_\beta$$
by $\beta - 1$ being even and using the inductive hypothesis.
- If β is an even successor, the calculation is similar, using that $g^0_{\beta-1} \geq_p g^1_{\beta-1}$ we obtain
$$g^0_\beta = f^0_{\beta-1} - g^0_{\beta-1} \leq_p f^0_{\beta-1} - g^1_{\beta-1} = f^1_{\beta-1} - g^1_{\beta-1} = g^1_\beta.$$

Consequently, the induction shows that $g^0_\delta \leq_p g^1_\delta$ if δ is even and $g^0_\delta \geq_p g^1_\delta$ if δ is odd. Therefore, since $\widehat{g^i_\delta} = f^i_\delta$ we have that $f^0_\delta \leq_p f^1_\delta$ if δ is even and $f^0_\delta \geq_p f^1_\delta$ if δ is odd. But by the definition of δ it is clear that $f^0_\delta \neq f^1_\delta$, hence $f^0_\delta <_p f^1_\delta$ if δ is even and $f^0_\delta >_p f^1_\delta$ if δ is odd. This finishes the proof of Proposition 3.2.6. □

Now to finish the proof of Theorem 3.2.1 we need the following folklore lemma.

Lemma 3.2.7. *There exists an order preserving embedding $\Psi_0 : USC^+(X) \hookrightarrow [0, 1]$ where the image of the function $f \equiv 0$ is 0. In particular, there is no uncountable strictly monotone transfinite sequence in $USC^+(X)$.*

Proof. Fix a countable basis $\{B_n : n \in \omega\}$ of $X \times [0, \infty)$. Assign to each $f \in USC^+$ the real
$$r_f = 1 - \sum_{B_n \cap sgr(f) = \emptyset} 2^{-n-1}.$$

If $f <_p g$ then $sgr(f) \subsetneq sgr(g)$ so, as the subgraph of an USC function is a closed set, there exists an $n \in \omega$ so that B_n is an open neighbourhood of a point in $sgr(g) \setminus sgr(f)$. Thus, $\{n : B_n \cap sgr(f) = \emptyset\} \supsetneq \{n : B_n \cap sgr(g) = \emptyset\}$. Consequently, $r_f < r_g$. □

Proof of Theorem 3.2.1. Let $\Psi : \sigma^* USC^+(X) \to \sigma^*[0,1]$ be the map that applies the above Ψ_0 to every coordinate of the sequences in $\sigma^* USC^+(X)$. Thus, Ψ is order preserving coordinate-wise.

Clearly, $h(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$ is an order preserving homeomorphism from $\mathbb{R}$ to $(0,1)$ and for $f \in \mathcal{B}_1(X)$ let $H(f) = h \circ f$. Composing the functions in $\mathcal{B}_1(X)$ with h we still have Baire class 1 functions and this does not effect the pointwise ordering. Thus, H is an order preserving map from $\mathcal{B}_1(X)$ into $b\mathcal{B}_1^+(X)$.

Let $\Theta = \Psi \circ \Phi \circ H$. Notice that as $H : \mathcal{B}_1(X) \to b\mathcal{B}_1^+(X)$, $\Phi : b\mathcal{B}_1^+(X) \to \sigma^* bUSC^+(X)$ and $\Psi : \sigma^* USC(X) \to \sigma^*[0,1]$, the map Θ is well defined.

Now, by Lemma 3.2.7 we have that Ψ_0 maps the constant zero function to zero and by (2) of Proposition 3.2.5 we have that for every function f its Φ image ends with the constant zero function. Thus, the Θ image of every function f ends with zero. Therefore, Θ maps into $[0,1]^{<\omega_1}_{\searrow 0}$.

If $f_0 <_p f_1$ are Baire class 1 functions then clearly $H(f_0) <_p H(f_1)$ hence by Proposition 3.2.6 we have that if $\delta = \delta(\Phi(H(f_0)), \Phi(H(f_1)))$, then $\Phi(H(f_0))(\delta) <_p \Phi(H(f_1))(\delta)$ if δ is even and $\Phi(H(f_0))(\delta) >_p \Phi(H(f_1))(\delta)$ if δ is odd. Since Ψ is order preserving coordinate-wise, we obtain that Θ is an order preserving embedding of $\mathcal{B}_1(X)$ into $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, which finishes the proof of the theorem. □

3.2.2 $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow \mathcal{B}_1(X)$

Theorem 3.2.8. *The linearly ordered set $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$ can be represented by $\mathbf{\Delta}^0_2$ subsets of $\mathcal{K}([0,1]^2)$ ordered by inclusion.*

Proof. First we define a map $\Psi : [0,1]^{<\omega_1}_{\searrow 0} \to \mathcal{K}([0,1]^2)$, basically assigning to each sequence its closure (as a subset of the interval). However, such a map cannot distinguish between continuous sequences and sequences omitting a limit point. To remedy this we place a line segment on each limit point contained in the sequence.

Let $\bar{x} \in [0,1]^{<\omega_1}_{\searrow 0}$, with $\bar{x} = (x_\alpha)_{\alpha \le \xi}$. Now let

$$\Psi(\bar{x}) = \overline{\{(x_\alpha, 0) : \alpha \le \xi\}} \cup$$

$$\bigcup \{\{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] : \text{ if } 0 < \alpha < \xi \text{ and } x_\alpha = \inf\{x_\beta : \beta < \alpha\}\}.$$

Lemma 3.2.9. *$\Psi(\bar{x})$ is a compact set for every $\bar{x} \in [0,1]^{<\omega_1}_{\searrow 0}$.*

Proof. Clearly, it is enough to show that if $(p_n, q_n) \to (p, q)$ is a convergent sequence such that for every n we have

$$(p_n, q_n) \in \bigcup\{\{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] : \text{ if } 0 < \alpha < \xi \text{ and } x_\alpha = \inf\{x_\beta : \beta < \alpha\}\} \tag{3.2.1}$$

then $(p, q) \in \Psi(\bar{x})$.

Obviously, $p_n = x_{\alpha_n}$ for some ordinals α_n. First, if the sequence x_{α_n} is eventually constant, then there exists an α so that $p = x_\alpha$ and except for finitely many n's by (3.2.1) we have $q_n \in [0, x_\alpha - x_{\alpha+1}]$. So $(p, q) \in \{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] \subset \Psi(\bar{x})$.

Now if the sequence $(x_{\alpha_n})_{n\in\omega}$ is not eventually constant, since the sequence $(x_\alpha)_{\alpha\leq\xi}$ is strictly decreasing and well-ordered then (passing to a subsequence of $(x_{\alpha_n})_{n\in\omega}$ if necessary) we can suppose that $(x_{\alpha_n})_{n\in\omega}$ is a strictly decreasing sequence.

Using the fact that $(x_{\alpha_n})_{n\in\omega}$ is a strictly decreasing subset of $(x_\alpha)_{\alpha\leq\xi}$ we obtain that $x_{\alpha_n} - x_{\alpha_n+1} \leq x_{\alpha_n} - p$. Hence from (3.2.1) we get

$$0 \leq q_n \leq x_{\alpha_n} - x_{\alpha_n+1} \leq x_{\alpha_n} - p \to 0$$

so $q_n = 0$. Therefore,

$$(p, q) = (\lim_{n\to\infty} x_{\alpha_n}, 0) \in \overline{\{(x_\alpha, 0) : \alpha \leq \xi\}} \subset \Psi(\bar{x}).$$

□

Now we define a decreasing sequence of subsets of $\mathcal{K}([0, 1]^2)$ for each $\bar{x} = (x_\alpha)_{\alpha\leq\xi}$ and $\alpha \leq \xi$ as follows:

$$\mathcal{H}^{\bar{x}}_\alpha = \{\Psi(\bar{z}) : \bar{z}|_\alpha = \bar{x}|_\alpha, z_\alpha \leq x_\alpha\}. \tag{3.2.2}$$

We will use the following notations for an even ordinal $\alpha \leq \xi$:

$$\mathcal{K}^{\bar{x}}_\alpha = \overline{\mathcal{H}^{\bar{x}}_\alpha} (= \overline{\{\Psi(\bar{z}) : \bar{z}|_\alpha = \bar{x}|_\alpha, z_\alpha \leq x_\alpha\}}), \tag{3.2.3}$$

and if $\alpha + 1 \leq \xi$ then

$$\mathcal{L}^{\bar{x}}_\alpha = \overline{\mathcal{H}^{\bar{x}}_{\alpha+1}} (= \overline{\{\Psi(\bar{z}) : \bar{z}|_{\alpha+1} = \bar{x}|_{\alpha+1}, z_{\alpha+1} \leq x_{\alpha+1}\}}). \tag{3.2.4}$$

Finally, if $\alpha = \xi$ then let $\mathcal{L}^{\bar{x}}_\alpha = \emptyset$. So $\mathcal{K}^{\bar{x}}_\alpha$ and $\mathcal{L}^{\bar{x}}_\alpha$ is defined for every even $\alpha \leq \xi$.

Notice that the sequence $(\overline{\mathcal{H}^{\bar{x}}_\alpha})_{\alpha\leq\xi}$ is a decreasing sequence of closed sets.

To each $\bar{x} = (x_\alpha)_{\alpha\leq\xi}$ let us assign

$$\mathcal{A}^{\bar{x}} = \bigcup_{\alpha\leq\xi, \alpha \text{ even}} (\mathcal{K}^{\bar{x}}_\alpha \setminus \mathcal{L}^{\bar{x}}_\alpha).$$

By [40, 22.27], since $\mathcal{A}^{\bar{x}}$ is a transfinite difference of a decreasing sequence of closed sets, we have $\mathcal{A}^{\bar{x}} \in \mathbf{\Delta}^0_2(\mathcal{K}([0, 1]^2))$.

To overcome some technical difficulties we prove the following lemma.

Lemma 3.2.10. *Let $\bar{z} \in [0,1]^{<\omega_1}_{\searrow 0}$ and β be an ordinal such that $\beta + 1 \leq l(\bar{z})$.*

(1) If $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta+1}}$, β is a limit ordinal, $\inf\{z_\gamma : \gamma < \beta\} = z_\beta$ and $l(\bar{z}) > \beta + 1$ then $(z_\beta, z_\beta - z_{\beta+1}) \in K$.

(2) If $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta}}$ and β is a successor then $(z_{\beta-1}, 0) \in K$.

(3) If $K \in \overline{\mathcal{H}^{\bar{z}}_{\beta}}$, β is a limit ordinal and $\inf\{z_\gamma : \gamma < \beta\} > z_\beta$ OR β is a successor then

$$K \cap ((z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0,1]) = \emptyset$$

(notice that if β is a successor then $\inf\{z_\gamma : \gamma < \beta\} = z_{\beta-1}$).

Proof. For (2) and (1) just notice that by equation (3.2.2) whenever $\Psi(\bar{w}) \in \mathcal{H}^{\bar{z}}_\beta$ ($\mathcal{H}^{\bar{z}}_{\beta+1}$, respectively) then $\Psi(\bar{w})$ contains the point $(z_{\beta-1}, 0)$ (the point $(z_\beta, z_\beta - z_{\beta+1})$). Consequently, every compact set which is in the closure of $\mathcal{H}^{\bar{z}}_\beta$ (or $\mathcal{H}^{\bar{z}}_{\beta+1}$) contains the point $(z_{\beta-1}, 0)$ (the point $(z_\beta, z_\beta - z_{\beta+1})$).

(3) can be proved similarly: by the definition of $\mathcal{H}^{\bar{z}}_\beta$ for every $\bar{w}$ such that $\Psi(\bar{w}) \in \mathcal{H}^{\bar{z}}_\beta$ we have

$$\Psi(\bar{w}) \cap ((z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0,1]) = \emptyset.$$

Now since the set $U = (z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0,1]$ is relatively open in $[0,1]^2$, the set $\{K \in \mathcal{K}([0,1]^2) : K \cap U = \emptyset\}$ is closed. Hence $\mathcal{H}^{\bar{z}}_\beta \subset \{K \in \mathcal{K}([0,1]^2) : K \cap U = \emptyset\}$ implies that every $K \in \overline{\mathcal{H}^{\bar{z}}_\beta}$ is disjoint from U. So we proved the lemma. □

In order to show that $\bar{x} \mapsto \mathcal{A}^{\bar{x}}$ is an embedding it is enough to prove the following claim.

Main Claim. If $\bar{x} <_{altlex} \bar{y}$ then $\mathcal{A}^{\bar{x}} \subsetneq \mathcal{A}^{\bar{y}}$.

To verify this we have to distinguish two cases.

Case 1. $\delta = \delta(\bar{x}, \bar{y})$ is even. Then $x_\delta < y_\delta$ and $\delta + 1 < l(\bar{y})$. We will show the following lemma.

Lemma 3.2.11. $\mathcal{K}^{\bar{x}}_\delta \subsetneq \mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta$.

Proof of Lemma 3.2.11. From $x_\delta < y_\delta$ we have

$$\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq x_\delta\} \subset \{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq y_\delta\}$$

so $\mathcal{K}^{\bar{x}}_\delta \subset \mathcal{K}^{\bar{y}}_\delta$.

First, we prove that

$$\mathcal{K}^{\bar{x}}_\delta \subset \mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta. \tag{3.2.5}$$

Here we have to separate two subcases.

Subcase 1. δ is a limit ordinal and $y_\delta = \inf\{y_\alpha : \alpha < \delta\}$.

On the one hand, using (1) of Lemma 3.2.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we obtain that for every $K \in \mathcal{L}^{\bar{y}}_\delta (= \overline{\mathcal{H}^{\bar{y}}_{\delta+1}})$ we have $(y_\delta, y_\delta - y_{\delta+1}) \in K$.

On the other hand, from (3) of Lemma 3.2.10 (with $\bar{z} = \bar{x}$ and $\beta = \delta$) we get that for every $K \in \mathcal{K}^{\bar{x}}_\delta (= \overline{\mathcal{H}^{\bar{x}}_\delta})$ we have $K \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0,1]) = \emptyset$. In particular, as $y_\delta \in (x_\delta, \inf\{x_\alpha : \alpha < \delta\})$, we have $(y_\delta, y_\delta - y_{\delta+1}) \notin K$. So we obtain $\mathcal{K}^{\bar{x}}_\delta \cap \mathcal{L}^{\bar{y}}_\delta = \emptyset$, hence by $\mathcal{K}^{\bar{x}}_\delta \subset \mathcal{K}^{\bar{y}}_\delta$ we have $\mathcal{K}^{\bar{x}}_\delta \subset \mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta$.

Subcase 2. δ is a limit and $y_\delta < \inf\{y_{\delta'} : \delta' < \delta\}$ or δ is a successor.

Using (2) of Lemma 3.2.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta + 1$) we obtain that every $K \in \mathcal{L}^{\bar{y}}_\delta (= \overline{\mathcal{H}^{\bar{y}}_{\delta+1}})$ contains the point $(y_\delta, 0)$. From (3) of Lemma 3.2.10 (with $\bar{z} = \bar{x}$, $\beta = \delta$) we have that for every $K \in \mathcal{K}^{\bar{x}}_\delta (= \overline{\mathcal{H}^{\bar{x}}_\delta})$ the set $K \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0,1])$ is empty. But $y_\delta \in (x_\delta, \inf\{x_\alpha : \alpha < \delta\})$ so $\mathcal{K}^{\bar{x}}_\delta \cap \mathcal{L}^{\bar{y}}_\delta = \emptyset$. This finishes the proof of equation (3.2.5).

Second, in order to prove $\mathcal{K}^{\bar{x}}_\delta \neq \mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta$ let $\bar{w}$ be such that $\bar{w}|_\delta = \bar{x}|_\delta$, $x_\delta, y_{\delta+1} < w_\delta < y_\delta$ and $w_{\delta+1} = 0$. Clearly, $\Psi(\bar{w}) \in \mathcal{K}^{\bar{y}}_\delta$.

By (3) of Lemma 3.2.10 (used for $\bar{z} = \bar{x}$ and $\beta = \delta$) we have that $\Psi(\bar{w}) \in \mathcal{K}^{\bar{x}}_\delta (= \overline{\mathcal{H}^{\bar{x}}_\delta})$ would imply $\Psi(\bar{w}) \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0,1]) = \emptyset$, but $(w_\delta, 0) \in (x_\delta, y_\delta) \times [0,1]$ and $\inf\{x_\alpha : \alpha < \delta\} = \inf\{y_\alpha : \alpha < \delta\} \geq y_\delta$ which is a contradiction. Hence $\Psi(\bar{w}) \notin \mathcal{K}^{\bar{x}}_\delta$.

Now we prove $\Psi(\bar{w}) \notin \mathcal{L}^{\bar{y}}_\delta$. Suppose the contrary, then using (3) of Lemma 3.2.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta + 1$) one can obtain that for every $K \in \mathcal{L}^{\bar{y}}_\delta (= \overline{\mathcal{H}^{\bar{y}}_{\delta+1}})$ the set $K \cap ((y_{\delta+1}, y_\delta) \times [0,1])$ is empty. But clearly $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((y_{\delta+1}, y_\delta) \times [0,1])$, a contradiction. So $\Psi(\bar{w}) \notin \mathcal{L}^{\bar{y}}_\delta$.

Thus, it follows that $\Psi(\bar{w}) \in (\mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta) \setminus \mathcal{K}^{\bar{x}}_\delta$. From this and from equation (3.2.5) we can conclude Lemma 3.2.11. □

Now we prove the Main Claim in Case 1. If δ' is even and $\delta' < \delta$, the definitions (3.2.3) and (3.2.4) of $\mathcal{K}^{\bar{y}}_{\delta'}$ and $\mathcal{L}^{\bar{y}}_{\delta'}$ depend only on $(x_\alpha)_{\alpha \leq \delta'+1}$ so

$$\mathcal{K}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \tag{3.2.6}$$

and

$$\mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{L}^{\bar{y}}_{\delta'}. \tag{3.2.7}$$

Now, from Lemma 3.2.11 we have $\mathcal{A}^{\bar{x}} \subset \mathcal{A}^{\bar{y}}$, since for every $K \in \mathcal{A}^{\bar{x}}$ we have either $K \in \mathcal{K}^{\bar{x}}_{\delta'} \setminus \mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \setminus \mathcal{L}^{\bar{y}}_{\delta'}$ for some $\delta' < \delta$ or $K \in \mathcal{K}^{\bar{x}}_\delta$.

Moreover, we claim that using Lemma 3.2.11 one can prove that $\mathcal{A}^{\bar{x}} \subsetneq \mathcal{A}^{\bar{y}}$. From the definition of $\mathcal{A}^{\bar{x}}$, from the fact that the sequence $(\mathcal{H}^{\bar{x}}_\alpha)_{\alpha \leq \xi} = (\mathcal{K}^{\bar{x}}_0, \mathcal{L}^{\bar{x}}_0, \mathcal{K}^{\bar{x}}_1, \mathcal{L}^{\bar{x}}_1, \dots)$ is decreasing and from equations (3.2.6) and (3.2.7) follows that

$$(\mathcal{K}^{\bar{x}}_\delta)^c \cap \mathcal{A}^{\bar{x}} = \bigcup_{\delta' < \delta,\ \delta' \text{ even}} \mathcal{K}^{\bar{x}}_{\delta'} \setminus \mathcal{L}^{\bar{x}}_{\delta'} = \bigcup_{\delta' < \delta,\ \delta' \text{ even}} \mathcal{K}^{\bar{y}}_{\delta'} \setminus \mathcal{L}^{\bar{y}}_{\delta'} = (\mathcal{K}^{\bar{y}}_\delta)^c \cap \mathcal{A}^{\bar{y}}$$

So $\mathcal{A}^{\bar{x}} \subset (\mathcal{K}^{\bar{y}}_\delta)^c \cup \mathcal{K}^{\bar{x}}_\delta$. Hence, if $K \in (\mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta) \setminus \mathcal{K}^{\bar{x}}_\delta$ then

$$K \in \mathcal{K}^{\bar{y}}_\delta \setminus \mathcal{L}^{\bar{y}}_\delta \subset \mathcal{A}^{\bar{y}}$$

and

$$K \notin (\mathcal{K}^{\bar{y}}_\delta)^c \cup \mathcal{K}^{\bar{x}}_\delta \supset \mathcal{A}^{\bar{x}}$$

so indeed, we obtain that the containment is strict, hence we are done with Case 1.

Case 2. $\delta = \delta(\bar{x}, \bar{y})$ is odd.
Then $x_\delta > y_\delta$ and $\delta + 1 < l(\bar{x})$. Notice that as the length of $\bar{x}$ is larger than $\delta + 1$, the sets $\mathcal{K}^{\bar{x}}_{\delta+1}$ and $\mathcal{L}^{\bar{x}}_{\delta+1}$ are defined.

Now for every even $\delta' \leq \delta - 1$ the definition of $\mathcal{K}^{\bar{x}}_{\delta'}$ and $\mathcal{K}^{\bar{y}}_{\delta'}$ depend only on $(x_\alpha)_{\alpha \leq \delta'} = (y_\alpha)_{\alpha \leq \delta'}$. Thus for every even $\delta' \leq \delta - 1$

$$\mathcal{K}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \tag{3.2.8}$$

and also for every even $\delta' < \delta - 1$

$$\mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{L}^{\bar{y}}_{\delta'}. \tag{3.2.9}$$

We will show the following:

Lemma 3.2.12. *(1)* $\mathcal{K}^{\bar{x}}_{\delta-1} \setminus \mathcal{L}^{\bar{x}}_{\delta-1} \subset \mathcal{K}^{\bar{y}}_{\delta-1} \setminus \mathcal{L}^{\bar{y}}_{\delta-1}$

(2) $\mathcal{K}^{\bar{x}}_{\delta+1} \subset \mathcal{K}^{\bar{y}}_{\delta-1} \setminus \mathcal{L}^{\bar{y}}_{\delta-1}$.

Proof of Lemma 3.2.12. It is easy to prove (1): from equation (3.2.8) we get $\mathcal{K}^{\bar{x}}_{\delta-1} = \mathcal{K}^{\bar{y}}_{\delta-1}$. Moreover, $\mathcal{L}^{\bar{x}}_{\delta-1} \supset \mathcal{L}^{\bar{y}}_{\delta-1}$, since

$$\mathcal{L}^{\bar{x}}_{\delta-1} = \overline{\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq x_\delta\}} \supset \overline{\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{y}|_\delta, z_\delta \leq y_\delta\}} = \mathcal{L}^{\bar{y}}_{\delta-1}$$

holds by $x_\delta > y_\delta$.

Now we show (2). First, $\mathcal{K}^{\bar{x}}_{\delta+1} \subset \mathcal{K}^{\bar{x}}_{\delta-1} = \mathcal{K}^{\bar{y}}_{\delta-1}$, using that the sequence $(\mathcal{K}^{\bar{x}}_\alpha)_{\alpha \leq \delta+1}$ is decreasing.

So it is suffices to show that $\mathcal{K}^{\bar{x}}_{\delta+1} \cap \mathcal{L}^{\bar{y}}_{\delta-1} = \emptyset$. Using (3) of Lemma 3.2.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we obtain that for every $K \in \mathcal{L}^{\bar{y}}_{\delta-1} (= \overline{\mathcal{H}^{\bar{y}}_\delta})$, we have $K \cap ((y_\delta, y_{\delta-1}) \times [0, 1]) = \emptyset$.
However, by (2) of Lemma 3.2.10 (used with $\bar{z} = \bar{x}$ and $\beta = \delta + 1$) if $K \in \mathcal{K}^{\bar{x}}_{\delta+1} (= \overline{\mathcal{H}^{\bar{x}}_{\delta+1}})$ then $(x_\delta, 0) \in K$. Therefore, $x_\delta \in (y_\delta, y_{\delta-1})$ implies that the intersection $\mathcal{K}^{\bar{x}}_{\delta+1} \cap \mathcal{L}^{\bar{y}}_{\delta-1}$ must be empty. So we are done with the lemma. □

Now we prove the Main Claim in Case 2. By definition of $\mathcal{A}^{\bar{x}}$ and by the fact that the sequence $(\mathcal{H}^{\bar{x}}_\alpha)_{\alpha \leq \xi} = (\mathcal{K}^{\bar{x}}_0, \mathcal{L}^{\bar{x}}_0, \mathcal{K}^{\bar{x}}_1, \mathcal{L}^{\bar{x}}_1, \dots)$ is decreasing we have that if $K \in \mathcal{A}^{\bar{x}}$ then either $K \in \mathcal{K}^{\bar{x}}_{\delta'} \setminus \mathcal{L}^{\bar{x}}_{\delta'} = \mathcal{K}^{\bar{y}}_{\delta'} \setminus \mathcal{L}^{\bar{y}}_{\delta'}$ for some even $\delta' < \delta - 1$ or $K \in \mathcal{K}^{\bar{x}}_{\delta-1} \setminus \mathcal{L}^{\bar{x}}_{\delta-1}$ or $K \in \mathcal{K}^{\bar{x}}_{\delta+1}$. Hence using equations (3.2.8) and (3.2.9) and Lemma 3.2.12 we obtain

$$\mathcal{A}^{\bar{x}} \subset \mathcal{A}^{\bar{y}}. \tag{3.2.10}$$

In order to show that $\mathcal{A}^{\bar{x}} \neq \mathcal{A}^{\bar{y}}$ it is enough to find a $\bar{w}$ such that

$$\Psi(\bar{w}) \in \mathcal{K}^{\bar{y}}_{\delta-1} \setminus \mathcal{L}^{\bar{y}}_{\delta-1} \subset \mathcal{A}^{\bar{y}} \tag{3.2.11}$$

and

$$\Psi(\bar{w}) \notin \mathcal{K}^{\bar{x}}_{\delta+1} \cup (\mathcal{L}^{\bar{x}}_{\delta-1})^c \supset \mathcal{A}^{\bar{x}}. \tag{3.2.12}$$

Take $\bar{w}|_\delta = \bar{y}|_\delta$ and w_δ such that $x_{\delta+1}, y_\delta < w_\delta < x_\delta$ and $w_{\delta+1} = 0$.

Now, in order to see (3.2.11) clearly $\Psi(\bar{w}) \in \mathcal{K}^{\bar{y}}_{\delta-1}$. On the other hand if $K \in \mathcal{L}^{\bar{y}}_{\delta-1}(= \overline{\mathcal{H}^{\bar{y}}_\delta})$ by (3) of Lemma 3.2.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we have $K \cap ((y_\delta, y_{\delta-1}) \times [0,1]) = \emptyset$. But $y_\delta < w_\delta < x_\delta < x_{\delta-1} = y_{\delta-1}$, so $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((y_\delta, y_{\delta-1}) \times [0,1])$. Therefore, $\Psi(\bar{w}) \notin \mathcal{L}^{\bar{y}}_{\delta-1}$.

In order to prove (3.2.12) it is obvious that $\Psi(\bar{w}) \in \mathcal{L}^{\bar{x}}_{\delta-1}$. Now using again (3) of Lemma 3.2.10 (with $\bar{z} = \bar{x}$ and $\beta = \delta + 1$) we obtain that whenever $K \in \mathcal{K}^{\bar{x}}_{\delta+1}(= \overline{\mathcal{H}^{\bar{x}}_{\delta+1}})$ then $K \cap ((x_{\delta+1}, x_\delta) \times [0,1]) = \emptyset$. However, $w_\delta \in (x_{\delta+1}, x_\delta)$ hence $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((x_{\delta+1}, x_\delta) \times [0,1])$, so $\Psi(\bar{w}) \notin \mathcal{K}^{\bar{x}}_{\delta+1}$.

So we can conclude that $\mathcal{A}^{\bar{x}} \neq \mathcal{A}^{\bar{y}}$. Thus, using equation (3.2.10) we can finish the proof of the Main Claim in Case 2 and hence we obtain Theorem 3.2.8 as well. □

3.2.3 The main theorem

Theorem 3.2.13. *(Main Theorem) Let X be an uncountable Polish space. Then the following are equivalent for a linear ordering $(L, <)$:*

(1) $(L, <) \hookrightarrow (\mathcal{B}_1(X), <_p)$

(2) $(L, <) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$

(3) $(L, <) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$

In fact, $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, $(\mathbf{\Delta}^0_2(X), \subsetneq)$ *and* $(\mathcal{B}_1(X), <_p)$ *are embeddable into each other.*

Proof. $(\mathcal{B}_1(X), <_p) \hookrightarrow ([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$: Theorem 3.2.1.

$([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$: we proved in Theorem 3.2.8 that $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(\mathcal{K}([0,1]^2)), \subsetneq)$. Now, [19, Theorem 1.2] states that the class of linear orderings representable in $\mathbf{\Delta}^0_2$ coincide for all uncountable σ-compact Polish spaces. Hence, if C is the Cantor space, then $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(C), \subsetneq)$. If X is an uncountable Polish space then there exists a continuous injection $h : C \to X$. Now, since $h(C)$ is a closed set in X we have that $A \mapsto h(A)$ is an inclusion-preserving embedding $(\mathbf{\Delta}^0_2(C), \subsetneq) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$. Consequently, $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex}) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$.

$(\mathbf{\Delta}^0_2(X), \subsetneq) \hookrightarrow (\mathcal{B}_1(X), <_p)$: if A is a $\mathbf{\Delta}^0_2$ set then χ_A is a Baire class 1 function and $A \mapsto \chi_A$ is an order preserving $(\mathbf{\Delta}^0_2(X), \subsetneq) \hookrightarrow (\mathcal{B}_1(X), <_p)$ map. □

We immediately obtain the answers to Questions 5.2 and 5.3 from [23].

Corollary 3.2.14. *There exists an embedding $\mathcal{B}_1(X) \hookrightarrow \mathbf{\Delta}^0_2(X)$, hence a linear ordering is representable by Baire class* 1 *functions iff it is representable by Baire class* 1 characteristic *functions.*

The equivalence of (1) and (2), implies that the embeddability of a linearly ordered set into the set of Baire class 1 functions does not depend on the underlying Polish space (provided of course that the Polish space is uncountable). This result answers Question 1.5 from [19] affirmatively.

Corollary 3.2.15. *If X and Y are uncountable Polish spaces and $L \hookrightarrow \mathcal{B}_1(X)$ then $L \hookrightarrow \mathcal{B}_1(Y)$.*

From now on we will simply use the notation $\mathcal{B}_1(X) = \mathcal{B}_1$.

3.3 New proofs of known theorems

In this section we would like to demonstrate the strength and applicability of our characterisation by providing new, simpler proofs of the theorems of Kuratowski, Komjáth, Elekes and Steprāns. In case of Komjáth's result our proof does not use the technique of forcing, which is an answer to a question of Laczkovich [46].

We would like to remark here that the above authors mainly investigated $\mathcal{B}_1(\mathbb{R})$ and $\mathcal{B}_1(\omega^\omega)$, but as we saw in Corollary 3.2.15 the statements do not depend on the underlying Polish space, so we will state them slightly more generally.

3.3.1 Kuratowski's theorem

Theorem 3.3.1. *(Kuratowski, [44, §24. III.2.]) ω_1 and ω_1^* are not representable in $\mathcal{B}_1$.*

Proof. By Theorem 3.2.13 it is enough to prove that $\omega_1 \not\hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$ and $\omega_1^* \not\hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. We will prove the former statement, the proof of the latter is the same.

Suppose that $(f_\alpha)_{\alpha<\omega_1}$ is a strictly increasing sequence in $[0,1]^{<\omega_1}_{\searrow 0}$. Now we define a sequence $\{s_\alpha : \alpha < \omega_1\} \subset \sigma^*[0,1]$ that is strictly increasing with respect to containment. Notice that this will yield a contradiction, since $\cup_{\alpha<\omega_1} s_\alpha$ would be an ω_1-long strictly decreasing sequence of reals.

We define the sequence s_α by induction on α with the following properties:

$$l(s_\alpha) = \alpha \text{ and } \{\gamma : s_\alpha \subset f_\gamma\} \text{ contains an end segment of } \omega_1. \tag{3.3.1}$$

First, $s_0 = \emptyset$ clearly works. Now suppose that we are done for every $\beta < \alpha$.

If α is a limit let $s_\alpha = \cup_{\beta<\alpha} s_\beta$. Then

$$\{\gamma : s_\alpha \subset f_\gamma\} = \bigcap_{\beta<\alpha} \{\gamma : s_\beta \subset f_\gamma\}$$

so the set $\{\gamma : s_\alpha \subset f_\gamma\}$ is the intersection of countably many sets that contain end segments, hence it contains an end segment. Therefore, (3.3.1) holds.

Let α be a successor and $S = \{\gamma : s_{\alpha-1} \subset f_\gamma\}$. If $\gamma, \gamma' \in S$ with $\gamma < \gamma'$ then clearly $f_\gamma <_{altlex} f_{\gamma'}$. By $s_{\alpha-1} \subset f_\gamma$, $s_{\alpha-1} \subset f_{\gamma'}$ and $l(s_{\alpha-1}) = \alpha - 1$ we obtain that $\delta(f_\gamma, f_{\gamma'}) \geq \alpha - 1$. So either $f_\gamma(\alpha-1) = f_{\gamma'}(\alpha-1)$ or $f_\gamma(\alpha-1) < f_{\gamma'}(\alpha-1)$ if $\alpha-1$ is even and $f_\gamma(\alpha-1) > f_{\gamma'}(\alpha-1)$ if $\alpha-1$ is odd. Therefore, $f_\gamma(\alpha-1) \leq f_{\gamma'}(\alpha-1)$ if $\alpha-1$ is even and $f_\gamma(\alpha-1) \geq f_{\gamma'}(\alpha-1)$ if $\alpha-1$ is odd. Consequently, the map $\gamma \mapsto f_\gamma(\alpha-1)$ is order preserving from S to the unit interval if $\alpha-1$ is even and order reversing if $\alpha-1$ is odd. But S contains an end segment by induction, and $[0,1]$ contains no subset of type ω_1 or ω_1^*, hence this map attains a constant value, say r on an end segment. Thus, $s_\alpha = s_{\alpha-1} {}^\frown r$ satisfies (3.3.1). □

3.3.2 Komjáth's theorem

Komjáth [43] has shown using forcing that a Suslin line is not representable in $\mathcal{B}_1(\mathbb{R})$. Laczkovich [46] asked if a forcing-free proof exists. Now we provide such a proof.

Theorem 3.3.2. *(Komjáth, [43]) A Suslin line is not representable in $\mathcal{B}_1$.*

Notation. Let $(T, <_T)$ be a tree. We denote by $T|_{succ}$ the set $\{t \in T : t \in Lev_\alpha(T), \alpha \text{ is a successor}\}$ ordered by the restriction of $<_T$. Notice that $T|_{succ}$ is also a tree, but it is not a subtree of T. If $t \in \sigma^*[0,1]$ we will use the notation I_t for the set $\{\bar{x} \in [0,1]^{<\omega_1}_{\searrow 0} : t \subset \bar{x}\}$.

Lemma 3.3.3. *Suppose that $\mathcal{S} \subset [0,1]^{<\omega_1}_{\searrow 0}$ is a nowhere separable Suslin line. Then $\sigma^*[0,1]|_{succ}$ contains a Suslin tree.*

Proof. Let

$$T = \{t \in \sigma^*[0,1] : |\mathcal{S} \cap I_t| \geq 2\}. \tag{3.3.2}$$

We claim that $(T, \subsetneq)$ is a Suslin tree.

First, T is clearly a subtree of $(\sigma^*[0,1], \subsetneq)$ and $\sigma^*[0,1]$ does not contain uncountable chains hence this is true for T as well.

Second, let $A \subset T$ be an antichain. Notice that for every pair of incomparable nodes $t, t' \in T$ the sets I_t and $I_{t'}$ are disjoint intervals of $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$, hence $I_t \cap \mathcal{S}$ and $I_{t'} \cap \mathcal{S}$ are also disjoint intervals in $\mathcal{S}$. By (3.3.2) these intervals are non-degenerate. Since $A \subset T$ is an antichain the set $\{I_t \cap \mathcal{S} : t \in A\}$ is a collection of pairwise disjoint non-empty intervals in $\mathcal{S}$. Using that $\mathcal{S}$ is nowhere separable for every t we can select a $J_t \subset I_t$ such that $\mathcal{S} \cap J_t$ is a non-empty open interval. By definition $\mathcal{S}$ is ccc so the set $\{J_t \cap \mathcal{S} : t \in A\}$ is countable. Hence A is countable, showing that T does not contain uncountable antichains.

Third, it is left to show that T is uncountable. Suppose the contrary. Notice first that for every $t \in T$ the set $\{r \in [0,1] : \mathcal{S} \cap I_{t^\frown r} \neq \emptyset\}$ is countable, otherwise, choosing points $\bar{p}_r \in \mathcal{S} \cap I_{t^\frown r}$ the map $r \mapsto \bar{p}_r$ would give an uncountable real subtype of $\mathcal{S}$, which is impossible (see [59, Proposition 3.5]). Hence, as T is also countable, we can select a countable subset D of $\mathcal{S}$ with the following property: for every $t \in T$ and $r \in [0,1]$ such that $\mathcal{S} \cap I_{t^\frown r} \neq \emptyset$ there exists a point $\bar{p} \in D$ such that $\bar{p} \in I_{t^\frown r}$.

We claim that D is dense in $\mathcal{S}$ which will contradict the non-separability of $\mathcal{S}$. In order to see this let $J \subset \mathcal{S}$ be a non-empty open interval. By passing to a subinterval of J (using that $\mathcal{S}$ is nowhere separable) we can assume that J is of the form $[\bar{x}, \bar{y}] \cap \mathcal{S}$ with $\bar{x} \neq \bar{y}$. Let $\bar{z} \in (\bar{x}, \bar{y}) \cap \mathcal{S}$ (such a $\bar{z}$ exists by the fact that $\mathcal{S}$ is nowhere separable). Clearly $\bar{x} <_{altlex} \bar{z} <_{altlex} \bar{y}$. Let $\delta_{\bar{x}} = \delta(\bar{x}, \bar{z})$ and $\delta_{\bar{y}} = \delta(\bar{y}, \bar{z})$. Then $l(\bar{z}) \geq \max\{\delta_{\bar{x}}, \delta_{\bar{y}}\} + 1$ and

$$\bar{x}(\delta_{\bar{x}}) < \bar{z}(\delta_{\bar{x}}) \iff \delta_{\bar{x}} \text{ even and } \bar{z}(\delta_{\bar{y}}) < \bar{y}(\delta_{\bar{y}}) \iff \delta_{\bar{y}} \text{ even.} \tag{3.3.3}$$

Suppose that $\delta_{\bar{x}} \geq \delta_{\bar{y}}$, the proof of the other case is the same. If $t = \bar{x} \cap \bar{z}$, then $\{\bar{x}, \bar{z}\} \subset I_t$, so by (3.3.2) we have $t \in T$. Clearly,

$$\bar{z} \in \mathcal{S} \cap I_{\bar{z}|_{\delta_{\bar{x}}+1}} = \mathcal{S} \cap I_{t^\frown \bar{z}(\delta_{\bar{x}})}$$

hence, by the definition of D we obtain that there exists a $\bar{p} \in D \cap I_{t^\frown \bar{z}(\delta_{\bar{x}})}$. We have $\bar{p}|_{\delta_{\bar{x}}+1} = \bar{z}|_{\delta_{\bar{x}}+1}$ so from $\delta_{\bar{x}} \geq \delta_{\bar{y}}$ we get

$$\delta(\bar{x}, \bar{p}) = \delta_{\bar{x}} \text{ and } \delta(\bar{y}, \bar{p}) = \delta_{\bar{y}},$$

moreover

$$\bar{p}(\delta_{\bar{x}}) = \bar{z}(\delta_{\bar{x}}) \text{ and } \bar{p}(\delta_{\bar{y}}) = \bar{z}(\delta_{\bar{y}}).$$

Therefore, using (3.3.3) we obtain that $\bar{x} <_{altlex} \bar{p} <_{altlex} \bar{y}$, so $\bar{p} \in D \cap (\bar{x}, \bar{y}) \subset D \cap J$. So D is a countable dense subset of $\mathcal{S}$, a contradiction.

This yields that T is uncountable, hence it is indeed a Suslin tree.

Finally, notice that T is a subtree of $\sigma^*[0,1]$ so $T|_{succ} \subset \sigma^*[0,1]|_{succ}$. Let $T' = T|_{succ}$. Clearly, T' is a subset of T and by definition the ordering of T' is the restriction of the ordering of T, so T' does not contain uncountable chains or antichains. In order to see that T' is uncountable first notice that the lengths of the elements in T are unbounded in ω_1, therefore the lengths of the elements on the successor levels are also unbounded. Hence T' is uncountable so T' is also a Suslin tree, which completes the proof of the lemma. □

For the sake of completeness we will prove the following classical facts about Suslin trees.

Lemma 3.3.4. *If D is a dense open subset of the Suslin tree T then $T \setminus D$ is countable.*

Proof. Let A be a maximal antichain in D. Clearly, A is countable. Let α be such that $\alpha > \sup\{l(s) : s \in A\}$. Now, if $\beta \geq \alpha$ arbitrary and $t \in Lev_\beta(T)$ then by the density of D there exists an $s_0 \in D$ such that $t \leq_T s_0$. From the facts that A is maximal and $\beta \geq \alpha$ we obtain that for some $s_1 \in A$ we have $s_1 \leq_T s_0$ and hence $s_1 \leq_T t$. But then, as D is open and $A \subset D$ we obtain that $t \in D$. This finishes the proof of the lemma. □

Lemma 3.3.5. *A Suslin tree is not $\mathbb{R}$-special.*

Proof. Suppose the contrary. Let T be a Suslin tree and $f : T \to \mathbb{R}$ be an order preserving map. We can suppose that $f(T)$ is a subset of $[0,1]$.

Let $n \in \omega$ and

$$D_n = \{t \in T : (\forall s \geq_T t)(f(s) \leq f(t) + \frac{1}{n+1})\}.$$

Clearly, D_n is open. We will show that it is also dense in T. In order to see this let $t_0 \in T$ be arbitrary. Then either $t_0 \in D_n$ or there exists an $t_1 \geq_T t_0$ such that $f(t_1) > f(t_0) + \frac{1}{n+1}$. Repeating this argument for t_1 we obtain either that $t_1 \in D_n$ or a $t_2 \geq_T t_1$ such that $f(t_2) > f(t_1) + \frac{1}{n+1} > f(t_0) + \frac{2}{n+1}$, etc. $f(T) \subset [0,1]$ implies that this procedure stops after at most $n+2$ steps, hence we obtain an $s \geq_T t_0$ such that $s \in D_n$. Therefore, the sets D_n are dense open subsets of T. By Lemma 3.3.4 the complement of $\cap_{n\in\omega} D_n$ is countable, hence there exists $s <_T t$ such that $s, t \in \cap_{n\in\omega D_n}$. But then clearly $f(t) = f(s)$, a contradiction. □

Now we are ready to prove the main result of this subsection.

Proof of Theorem 3.3.2. Suppose the contrary and let $\mathcal{S}'$ be a subset of $\mathcal{B}_1$ order isomorphic to a Suslin line. By Theorem 3.2.13 there exists an embedding $\Phi_0 : \mathcal{S}' \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. For $p, q \in \mathcal{S}'$ let $p \sim q$ if the interval $[p,q]$ is separable. Then $\sim$ is an equivalence relation and $\mathcal{S} = \mathcal{S}'/\sim$ is a nowhere separable Suslin line (for the details see [59, Section 3.]). For every $\sim$ equivalence class $[\cdot]$ fix a representative $p \in \mathcal{S}'$. It is easy to see that every equivalence class is an interval, so the map $\Phi([p]) = \Phi_0(p)$ is an order preserving embedding of $\mathcal{S}$ into $[0,1]^{<\omega_1}_{\searrow 0}$.

Now we can use Lemma 3.3.3 for $\Phi(\mathcal{S})$. This yields that there exists a Suslin tree $T \subset \sigma^*[0,1]|_{succ}$. Assign to each $t \in T$ the last element of t, namely, let $f(t) = t(l(t)-1)$.

Let $s, t \in T$ such that $s <_T t$. Then, as $s \neq t$, the sequences s and t are strictly decreasing and (using that $s <_T t \iff s \subsetneq t$) t is an end extension of s we obtain that $f(t) < f(s)$. Therefore, the map $1 - f$ is a strictly monotone map from the Suslin tree T to $\mathbb{R}$. This contradicts Lemma 3.3.5. □

3.3.3 Ordered sets of cardinality $< \mathfrak{c}$ and Martin's Axiom

In this subsection we reprove the results of Elekes and Steprāns from [23]. To formulate the statements, we need some preparation.

Suppose that $(L, <_L)$ is a linearly ordered set. A partition tree T_L of L is defined as follows: the elements of T_L are certain non-empty open intervals of L ordered by reverse inclusion. T_L is constructed by induction. Let $Lev_0(T_L) = \{L\}$.

Suppose that for an ordinal α we have defined $Lev_\beta(T_L)$ for all $\beta < \alpha$. If α is a successor, for every $I \in Lev_{\alpha-1}(T_L)$ fix non-empty intervals I_0 and I_1 such that $I_0 \cup I_1 = I$ and $I_0 \cap I_1 = \emptyset$ if such I_0, I_1 exist. Let

$$Lev_\alpha(T_L) = \bigcup\{I_0, I_1 : I \in Lev_{\alpha-1}(T_L)\}.$$

Now if α is a limit ordinal let

$$Lev_\alpha(T_L) = \{\bigcap_{\beta<\alpha} I_\beta : I_\beta \in Lev_\beta(T_L), \cap_{\beta<\alpha} I_\beta \neq \emptyset\}.$$

Somewhat ambiguously if $t \in T_L$ we will denote the corresponding interval of L by N_t.

We first verify the next proposition, which is interesting in its own right.

Proposition 3.3.6. *Let L be a linear ordering such that T_L, a partition tree of L is $\mathbb{R}$-special. Then $L \hookrightarrow \mathcal{B}_1$.*

Proof. Without loss of generality we can suppose that we have a strictly decreasing map $\Phi : T_L \to (0,1)$.

Lemma 3.3.7. *There exists a map $\Psi_0 : T_L \to \sigma^*[0,1]$ with the following properties for every $t, s \in T_L$:*

(1) if $s \leq_{T_L} t$ then $\Psi_0(s) \subset \Psi_0(t)$,

(2) if $N_s <_L N_t$ then $\Psi_0(s) <_{altlex} \Psi_0(t)$,

(3) $\inf \Psi_0(t) \geq \Phi(t)$.

Proof. We define Ψ_0 inductively on the levels of T_L. Suppose that we are done for every $\beta < \alpha$.

If α is a limit ordinal and $t \in Lev_\alpha(T_L)$, let

$$\Psi_0(t) = \bigcup_{t' <_{T_L} t} \Psi_0(t'). \tag{3.3.4}$$

Now let α be a successor ordinal. First notice that for every $t \in Lev_\alpha(T_L)$ by the fact that Φ is strictly decreasing and the inductive hypothesis for $t|_\alpha$ we have

$$\Phi(t) < \Phi(t|_\alpha) \leq \inf \Psi_0(t|_\alpha). \tag{3.3.5}$$

Let

$$A = \{t \in Lev_\alpha(T_L) : (\exists s \in Lev_\alpha(T_L))(s \neq t \wedge t|_\alpha = s|_\alpha)\}.$$

Now, if $t \notin A$ then using (3.3.5) there exists an $r \in [0,1]$ such that

$$\Phi(t) < r < \inf \Psi_0(t|_\alpha). \tag{3.3.6}$$

So let

$$\Psi_0(t) = \Psi_0(t|_\alpha)^\frown r. \tag{3.3.7}$$

Notice that if $t \in A$ then there exists exactly one $s \neq t$ such that $s \in Lev_\alpha(T_L)$ and $t|_\alpha = s|_\alpha$. Hence A is the union of pairs $\{s,t\}$ such that $s,t \in Lev_\alpha(T_L)$ and $t \neq s$ and $t|_\alpha = s|_\alpha$. We will define $\Psi_0(s)$ and $\Psi_0(t)$ simultaneously for such pairs. Since s and t

are incomparable, the intervals N_s and N_t are disjoint, so either $N_s <_L N_t$ or $N_s >_L N_t$. Using (3.3.5) and $s|_\alpha = t|_\alpha$ we obtain

$$\Phi(t), \Phi(s) < \Phi(t|_\alpha) \leq \inf \Psi_0(t|_\alpha).$$

From this it follows that we can choose $r, q \in (0, 1)$ such that

$$\Phi(t), \Phi(s) < r, q < \inf \Psi_0(t|_\alpha) \tag{3.3.8}$$

and

$$N_s <_L N_t \iff \Psi_0(t|_\alpha) ^\frown q <_{altlex} \Psi_0(t|_\alpha) ^\frown r, \tag{3.3.9}$$

so let

$$\Psi_0(t) = \Psi_0(t|_\alpha) ^\frown r \text{ and } \Psi_0(s) = \Psi_0(t|_\alpha) ^\frown q = \Psi_0(s|_\alpha) ^\frown q. \tag{3.3.10}$$

Thus, we have defined Ψ_0 on $Lev_\alpha(T_L)$ (first on the complement of A then on A as well). We claim that Ψ_0 satisfies properties (1)-(3).

We check (1). Let $s <_{T_L} t$ and $t \in Lev_\alpha(T_L)$. If α is a limit ordinal then by (3.3.4) clearly $\Psi_0(s) \subset \Psi_0(t)$. If α is a successor then $s \leq_{T_L} t|_\alpha$, hence from the inductive hypothesis and from equations (3.3.6) and (3.3.10) we obtain (1).

In order to prove (2) let s and t be given with $N_s <_L N_t$. If $s|_\alpha = t|_\alpha$ then $s, t \in Lev_\alpha(T_L)$ and α is a a successor. Then by equations (3.3.9) and (3.3.10) clearly (2) holds. If $s|_\alpha \neq t|_\alpha$ then there exists an ordinal $\beta < \alpha$, $s' \subset s$ and $t' \subset t$ such that $s', t' \in Lev_\beta(T_L)$ and $N_{s'} < N_{t'}$. Hence from the inductive hypothesis $\Psi_0(s') <_{altlex} \Psi_0(t')$ so from property (1) we have $\Psi_0(s) <_{altlex} \Psi_0(t)$.

Finally, in order to see (3) if α is a limit just notice that $\Phi(t) \leq \Phi(t')$ whenever $t' \leq_{T_L} t$ so by the inductive hypothesis we have

$$\Phi(t) \leq \inf_{t' <_{T_L} t} \Phi(t') \leq \inf_{t' <_{T_L} t} (\inf \Psi_0(t')) = \inf \Psi_0(t).$$

If α is a successor then for $t \notin A$ by (3.3.6) and (3.3.7), while for $t \in A$ by (3.3.8) and (3.3.10) we get (3).

Thus the induction works, so we have proved that such a Ψ_0 exists. □

Now we define the embedding $L \hookrightarrow [0, 1]^{<\omega_1}_{\searrow 0}$. For $x \in L$ let

$$\Psi(x) = \Big(\bigcup_{t \in T_L,\, x \in N_t} \Psi_0(t)\Big) ^\frown 0.$$

By the definition of a partition tree, if for s and t we have $x \in N_t \cap N_s$ then s and t are $\leq_{T_L}$-comparable. Hence by property (1) of Ψ_0 for every $x \in L$ we have $\Psi_0(x) \in \sigma^*[0, 1]$. Moreover, by $ran(\Phi) \subset (0, 1)$ and by property (3) we have that concatenating $\bigcup_{t \in T_L,\, x \in N_t} \Psi_0(t)$ with zero will give an element in $[0, 1]^{<\omega_1}_{\searrow 0}$.

We claim that the map Ψ is order preserving between $(L, <_L)$ and $([0, 1]^{<\omega_1}_{\searrow 0}, <_{altlex})$. Let $x, y \in L$ with $x <_L y$. Then there exist $s, t \in T_L$ such that $x \in N_s$ and $y \in N_t$ and $N_s <_L N_t$. Then by property (2) of Ψ_0 we have $\Psi_0(s) <_{altlex} \Psi_0(t)$. Therefore, $\Psi_0(s) \subset \Psi(x)$ and $\Psi_0(t) \subset \Psi(y)$ implies $\Psi(x) <_{altlex} \Psi(y)$. □

Theorem 3.3.8. *(MA) If L is a linearly ordered set of cardinality $< \mathfrak{c}$ then L is representable in $\mathcal{B}_1$ iff L does not contain ω_1 or ω_1^*.*

Proof. Let T_L be a partition tree of L. We claim that T_L does not contain uncountable chains. Suppose the contrary, let $\{t_\alpha : \alpha < \omega_1\} \subset T_L$ be a chain. Then N_{t_α} (denoted by N_α later on) is a strictly decreasing sequence of intervals in L. Therefore, for every α there exists an $x_\alpha \in N_\alpha \setminus N_{\alpha+1}$ such that either $N_{\alpha+1} <_L \{x_\alpha\}$ or $N_{\alpha+1} >_L \{x_\alpha\}$. Without loss of generality we can suppose that the set $R = \{\alpha : (\exists x_\alpha \in N_\alpha \setminus N_{\alpha+1})(N_{\alpha+1} <_L \{x_\alpha\})\}$ is uncountable. But then the sequence $(x_\alpha)_{\alpha\in R}$ is strictly decreasing in L and R is unbounded in ω_1 so $(x_\alpha)_{\alpha\in R}$ is order isomorphic to ω_1^*.

Notice that as every level of T_L contains pairwise disjoint non-empty intervals of L, from $|L| < \mathfrak{c}$ it follows that the cardinality of every level is strictly less than $\mathfrak{c}$. Moreover, since T_L does not contain uncountable chains, using that under Martin's Axiom $\mathfrak{c}$ is a regular cardinal we obtain that $|T_L| < \mathfrak{c}$.

Now it is easy to prove the theorem using a result of Baumgartner, Malitz and Reinhardt (see [6]) which states that assuming Martin's Axiom every tree with cardinality $< \mathfrak{c}$ that does not contain ω_1-chains is $\mathbb{Q}$-special. We have seen that T_L does not contain uncountable chains and $|T_L| < \mathfrak{c}$, hence it is $\mathbb{Q}$-special (in particular $\mathbb{R}$-special), so by Proposition 3.3.6 we have $L \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. By Theorem 3.2.13 this implies $L \hookrightarrow \mathcal{B}_1$. □

3.4 New results

3.4.1 Countable products and gluing

In this section we will answer Questions 2.2, 2.5 and 3.10 from [19]. Concerning the last question we would like to point out that in fact it has been already solved in [23].

Elekes [19] investigated several operations on collections of linearly ordered sets, and asked whether the closure of a simple collection of orderings under these operations coincide with the linearly ordered subsets of $\mathcal{B}_1$. We will first prove that the set of linearly ordered subsets of $\mathcal{B}_1$ is closed under the application of these operations.

Definition 3.4.1. Let L be a linearly ordered set and for every $p \in L$ fix a linearly ordered set L_p. Then the set $\{(p, q) : p \in L, q \in L_p\}$ ordered lexicographically (that is, $(p, q) <_g (p', q')$ if and only if $p <_L p'$ or $p = p'$ and $q <_{L_p} q'$) is called the *gluing of the L_p's along L.*

Theorem 3.4.2. *(1) Let $\{L_\beta : \beta < \alpha\}$ be a countable collection of linearly ordered sets that are representable in $\mathcal{B}_1$. Then the set $\prod_{\beta<\alpha} L_\beta$ ordered lexicographically is also representable.*

(2) Suppose that L and every $(L_p)_{p\in L}$ is representable in $\mathcal{B}_1$. Then the gluing of L_p's along L is also representable in $\mathcal{B}_1$.

Notation. Throughout this section if $\bar{x} = (x_\alpha)_{\alpha \leq \xi}$ is a transfinite sequence of reals and $a, b \in \mathbb{R}$ we will abbreviate the sequence $(ax_\alpha + b)_{\alpha \leq \xi}$ by $a\bar{x} + b$.

First we need a technical lemma.

Lemma 3.4.3. *Suppose that L is a linearly ordered set and there exists an embedding $\Psi : L \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. Then there exists an embedding $\Psi' : L \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$ such that for every $p \in L$ the length $l(\Psi'(p))$ is an even ordinal.*

Proof. It is easy to see that

$$\Psi'(p) = \begin{cases} (\frac{1}{2}\Psi(p) + \frac{1}{2}) ^\frown 0 & \text{if } l(\Psi(p)) \text{ is odd} \\ (\frac{1}{2}\Psi(p) + \frac{1}{2}) ^\frown \frac{1}{4} ^\frown 0 & \text{if } l(\Psi(p)) \text{ is even} \end{cases}$$

is also order preserving and takes every point $p \in L$ to a sequence with even length. □

Proof of Theorem 3.4.2. First we prove (1). The representability of L_β for every $\beta < \alpha$ by Theorem 3.2.13 imply that there exist embeddings $\Psi_\beta : L_\beta \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. Using Lemma 3.4.3 we can suppose that for every $\beta < \alpha$ and $p \in L_\beta$ the length of $\Psi_\beta(p)$ is even.

Fix now a sequence $(y_\beta)_{\beta \leq \alpha} \in \sigma^*[\frac{1}{2}, 1]$. For $\bar{p} = (p_\beta)_{\beta < \alpha} \in \prod_{\beta < \alpha} L_\beta$ let

$$\Psi(\bar{p}) = ({}^\frown_{\beta<\alpha}(\frac{y_\beta - y_{\beta+1}}{2}\Psi_\beta(p_\beta) + y_{\beta+1})) ^\frown 0,$$

where ${}^\frown_{\beta<\alpha}$ denotes concatenation of the sequences in type α.

We claim that Ψ is an embedding of $(\prod_{\beta<\alpha} L_\beta, <_{lex})$ into $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$. It is easy to see that for every $\bar{p} \in \prod_{\beta<\alpha} L_\beta$ we have $\Psi(\bar{p}) \in [0,1]^{<\omega_1}_{\searrow 0}$.

Now we prove that Ψ is order preserving. Let $\bar{p} <_{lex} \bar{q}$ with $\bar{p} = (p_\beta)_{\beta<\alpha}$, $\bar{q} = (q_\beta)_{\beta<\alpha}$ and let $\delta = \delta(\bar{p}, \bar{q})$, then $p_\delta <_{L_\delta} q_\delta$. It is easy to see that

$$\delta(\Psi(\bar{p}), \Psi(\bar{q})) = \sum_{\beta<\delta} l(\Psi_\beta(p_\beta)) + \delta(\Psi_\delta(p_\delta), \Psi_\delta(q_\delta)).$$

In particular, since every length in the previous equation is even we get that the $\delta(\Psi(\bar{p}), \Psi(\bar{q}))$ and $\delta(\Psi_\delta(p_\delta), \Psi_\delta(q_\delta))$ are of the same parity. Using this, $p_\delta <_{L_\delta} q_\delta$ and the fact that Ψ_δ is order preserving, we obtain that $\Psi(\bar{p}) <_{altlex} \Psi(\bar{q})$, which finishes the proof of (1).

(2) can be proved similarly. Fix an order preserving embedding $\Psi_0 : L \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$ such that for every $p \in L$ we have that $l(\Psi(p))$ is even. For every $p \in L$ let us also fix embeddings $\Psi_p : L_p \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. Then

$$\Psi(p, q) = (\frac{1}{2}(\Psi_0(p)) + \frac{1}{2}) ^\frown (\frac{1}{8}(\Psi_p(q)) + \frac{1}{4}) ^\frown 0$$

works. □

Definition 3.4.4. Let L be a linearly ordered set. The set $L \times 2$ ordered lexicographically is called the *duplication of* L.

Corollary 3.4.5. *A linearly ordered set is representable in* $\mathcal{B}_1$ *then its duplication is also representable.*

The first part of Theorem 3.4.2 answers Question 2.5, while Corollary 3.4.5 answers Question 2.2 from [19] affirmatively.

Now let us define the above mentioned operations on collections of linearly ordered sets. Suppose that $\mathcal{H}$ is an arbitrary set of ordered sets.

Definition 3.4.6. Let $\alpha < \omega_1$ be an ordinal, then

$$\mathcal{H}^\alpha = \{L_1 \subset L^\alpha : L \in \mathcal{H}\},$$

where L^α is ordered lexicographically. Let us denote by $\mathcal{H}^*$ the closure of $\mathcal{H}$ under the operation $\mathcal{H} \mapsto \mathcal{H}^\alpha$ for every $\alpha < \omega_1$.

Definition 3.4.7. $\mathcal{S}(\mathcal{H})$ denotes the closure of $\mathcal{H}$ under gluing.

It can be shown that such $\mathcal{H}^*$ and $\mathcal{S}(\mathcal{H})$ exist.

Suppose that every element of $\mathcal{H}$ is representable in $\mathcal{B}_1$. The first part of Theorem 3.4.2 clearly implies that every element of $\mathcal{H}^*$, while the second part yields that every element of $\mathcal{S}(\mathcal{H})$ is representable in $\mathcal{B}_1$. So it is natural to ask the following:

Question 3.4.8. *(Elekes, [19, Question 3.10.]) Does* $S(\{[0,1]^\alpha : \alpha < \omega_1\})^\omega$ *or* $S(\{[0,1]^\alpha : \alpha < \omega_1\})^*$ *equal to the linearly ordered sets representable in* $\mathcal{B}_1$*?*

To answer this question we need a property that is invariant under the above operations.

Definition 3.4.9. We say that a linearly ordered set L has property (*) if every uncountable subset of L contains an uncountable subset order-isomorphic to a subset of $\mathbb{R}$.

Proposition 3.4.10. *Suppose that every* $L \in \mathcal{H}$ *has property (*). Then (*) holds for every element of* $\mathcal{H}^*$ *and* $\mathcal{S}(\mathcal{H})$ *as well.*

Proof. In order to prove that every element of $\mathcal{H}^*$ has the required property it is enough to prove that if $\alpha < \omega_1$ and L has property (*) then so does L^α.

We prove this by induction on α. Suppose that we are done for every $\beta < \alpha$ and let $L_1 \subset L^\alpha$ be uncountable.

Observe that if there exists an ordinal $\beta < \alpha$ such that $L_2 = \{\bar{p} \in L^\beta : (\exists \bar{q})(\bar{p} ^\frown \bar{q} \in L_1)\}$ is uncountable then using that $L_2 \subset L^\beta$ and the inductive hypothesis we obtain that L_2 contains an uncountable real order type R_2. Thus, there exists an $R_1 \subset L_1$ such that for every $\bar{p} \in R_2$ there exists a unique $\bar{q}$ so that $\bar{p} ^\frown \bar{q} \in R_1$. It is easy to see that since L^α is ordered lexicographically we have that R_1 is an uncountable real order type in L_1 (in fact it is isomorphic to R_2).

So we can suppose that there is no such a β.

If α is a successor then using the above observation for $\beta = \alpha - 1$ we obtain that the set $\{\bar{p} \in L^{\alpha-1} : (\exists q \in L)(\bar{p}^\frown q \in L_1)\}$ is countable. By the uncountability of L_1 there exists a $\bar{p} \in L^{\alpha-1}$ such that the set $\{q : \bar{p}^\frown q \in L_1\}$ is uncountable. But this is a subset of L, so by the assumption on L there exists an uncountable real order type $R \subset \{q : \bar{p}^\frown q \in L_1\}$. Then $\{\bar{p}^\frown q : q \in R\}$ is an uncountable real order type in L_1.

Suppose now that α is a limit ordinal. By the above observation for every $\beta < \alpha$ the set $\{\bar{p} \in L^\beta : (\exists \bar{q})(\bar{p}^\frown \bar{q} \in L_1)\}$ is countable. So there exist countable sets $D_\beta \subset L_1$ with the following property: whenever for a point $\bar{p} \in L^\beta$ there exists a $\bar{q}$ such that $\bar{p}^\frown \bar{q} \in L_1$ then there exists a $\bar{q}'$ such that $\bar{p}^\frown \bar{q}' \in D_\beta$. Let $D = \bigcup_{\beta<\alpha} D_\beta$, then D is a countable set.

We claim that D is dense in L_1 (equipped with the order topology). In order to prove this let $\bar{x}, \bar{y} \in L_1$ such that $(\bar{x}, \bar{y}) \cap L_1$ is non-empty. Choose a $\bar{z} \in (\bar{x}, \bar{y}) \cap L_1$. Since α is a limit there exists a $\beta < \alpha$ such that $\beta > \max\{\delta(\bar{x}, \bar{z}), \delta(\bar{y}, \bar{z})\}$. Then there exists a $\bar{w} \in D_\beta \subset D$ such that $\bar{w}|_\beta = \bar{z}|_\beta$. But then clearly $\bar{w} \in (\bar{x}, \bar{y}) \cap L_1 \cap D$. So D is indeed dense. Consequently, L_1 contains an uncountable real order type (see [59, 3.2. Corollary]). This proves that L^α has property (*), so it is true for every element of $\mathcal{H}^*$.

In order to prove that every element of $\mathcal{S}(\mathcal{H})$ has property (*) one can use similar ideas: just use the above observation and the same argument as in the case of successor α. □

Now we are ready to answer Question 3.4.8. An *Aronszajn line* is an uncountable linearly ordered set that does not contain ω_1, ω_1^* and uncountable sets isomorphic to a subset of $\mathbb{R}$. An Aronszajn line is called *special* if it has an $\mathbb{R}$-special partition tree. Special Aronszajn lines exist, see [59, Theorem 5.1, 5.2]. Notice that Proposition 3.3.6 immediately gives the following important corollary:

Corollary 3.4.11. *If A is a special Aronszajn line then $A \hookrightarrow \mathcal{B}_1$.*

This corollary was proved by Elekes and Steprāns. Although it is not mentioned explicitly in the Elekes-Steprāns paper, the embeddability of the Aronszajn line answers the questions of Elekes negatively: on the one hand an Aronszajn line does not contain uncountable real order types. On the other hand by Proposition 3.4.10 every element of every collection of linear orderings obtainable from $\{[0,1]\}$ by the operations $\mathcal{H} \mapsto \mathcal{H}^*$ or $\mathcal{H} \mapsto \mathcal{S}(\mathcal{H})$ has property (*).

3.4.2 Completion

Now we will answer Question 2.7 from [19] negatively.

Theorem 3.4.12. *There exists a linearly ordered set such that it is representable in $\mathcal{B}_1$, but none of its completions are representable.*

Proof. Let $L \supset [0,1]^{<\omega_1}_{\searrow 0}$ be a completion of $[0,1]^{<\omega_1}_{\searrow 0}$, that is, a complete linear order containing $[0,1]^{<\omega_1}_{\searrow 0}$ as a dense subset. If it was representable then by Corollary 3.4.5

there would be an order preserving embedding $\Psi : L \times 2 \hookrightarrow [0,1]^{<\omega_1}_{\searrow 0}$. We will denote the lexicographical ordering on $L \times 2$ by $<_{L\times 2}$ and somewhat ambiguously the lexicographical ordering on $[0,1]^{<\omega_1}_{\searrow 0} \times 2$ by $<_{altlex \times 2}$. Notice that $<_{altlex\times 2}$ is the restriction of $<_{L\times 2}$ to $[0,1]^{<\omega_1}_{\searrow 0} \times 2$.

NOTATION. For each $s \in \sigma^*[0,1]$ let J_s be the basic interval in $[0,1]^{<\omega_1}_{\searrow 0} \times 2$ assigned to s, that is, the set $\{\bar{x} \in [0,1]^{<\omega_1}_{\searrow 0} : s \subset \bar{x}\} \times 2$. We will use the notation

$$I(s) = \Psi(\inf(J_s)) \text{ and } S(s) = \Psi(\sup(J_s)). \tag{3.4.1}$$

Notice that if L is complete then the set $L \times 2$ ordered lexicographically is also a complete linearly ordered set, hence $I(s)$ and $S(s)$ exist for every $s \in \sigma^*[0,1]$.

Let us define a map $\Phi : \sigma^*[0,1] \to [0,1]$ as follows:

Definition 3.4.13. For $s \in \sigma^*[0,1]$ let

$$\delta_s = \delta(I(s), S(s))$$

and

$$\Phi(s) = \max\{I(s)(\delta_s), S(s)(\delta_s)\}.$$

Let us also use the notation

$$\phi(s) = \min\{I(s)(\delta_s), S(s)(\delta_s)\}.$$

Notice that Φ and ϕ are well defined, since for every $s \in \sigma^*[0,1]$ the interval J_s contains at least two elements (one with last element 0 and another with 1), so $I(s)$ and $S(s)$ must differ. From this we have for all s that

$$0 \leq \phi(s) < \Phi(s). \tag{3.4.2}$$

In the following lemma we collect the easy observations that will be needed in the proof of the theorem.

Lemma 3.4.14. *Let $s, t, u \in \sigma^*[0,1]$ with $s \subset t$. Then*

(1) $\delta_s \leq \delta_t$,

(2) *(a)* $\Phi(s) \geq \Phi(t)$,

(b) $\max\{I(t)(\delta_s), S(t)(\delta_s)\} \leq \Phi(s)$,

(3) if $\delta \leq \delta_t$ *then* $\Phi(t) \leq \max\{I(t)(\delta), S(t)(\delta)\}$,

(4) if $\Phi(s) = \Phi(t)$ *then* $\delta_s = \delta_t$,

(5) if $r, q \in [0,1]$ *such that* $t^\frown r \leq_{altlex} t^\frown q$ *then*

(a) $I(t^\frown r)|_{\delta_t} = S(t^\frown r)|_{\delta_t} = I(t^\frown q)|_{\delta_t} = S(t^\frown q)|_{\delta_t}$,

(b) $I(t^\frown r)(\delta_t) \leq S(t^\frown r)(\delta_t) \leq I(t^\frown q)(\delta_t) \leq S(t^\frown q)(\delta_t)$ *if* δ_t *is even,*

(c) $I(t^\frown r)(\delta_t) \geq S(t^\frown r)(\delta_t) \geq I(t^\frown q)(\delta_t) \geq S(t^\frown q)(\delta_t)$ *if* δ_t *is odd,*

(6) if $t \leq_{altlex} u$ *and* δ *is an even ordinal such that* $I(t)|_\delta = S(t)|_\delta = I(u)|_\delta$ *then*

$$I(t)(\delta) \leq S(t)(\delta) \leq I(u)(\delta).$$

Proof. $J_s \supset J_t$, so by the fact that Ψ is order preserving we get

$$I(s) \leq_{altlex} I(t) \leq_{altlex} S(t) \leq_{altlex} S(s).$$

Therefore, by the definition of $<_{altlex}$ it is clear that $\delta_s \leq \delta_t$, so we have (1).

Now we show part (b) of (2). It is easy to see from the definition of $<_{altlex}$ that for every $\bar{x} \in [\inf(J_s), \sup(J_s)]$ we have $\Psi(\bar{x})(\delta_s) \in [\phi(s), \Phi(s)]$. In particular, as $[\inf(J_t), \sup(J_t)] \subset [\inf(J_s), \sup(J_s)]$ we obtain

$$\max\{I(t)(\delta_s), S(t)(\delta_s)\} \in [\phi(s), \Phi(s)], \tag{3.4.3}$$

which gives part (b). Since $I(t)$ and $S(t)$ are strictly decreasing sequences, using (1) we have

$$I(t)(\delta_t) \leq I(t)(\delta_s) \text{ and } S(t)(\delta_t) \leq S(t)(\delta_s).$$

Hence, (3.4.3) yields that $\Phi(t) \leq \Phi(s)$. Thus we have verified (2).

In order to see (3), use again that the sequences $I(t)$ and $S(t)$ are decreasing. Hence from $\delta \leq \delta_t$ and the definition of δ_t we have (3):

$$\Phi(t) = \max\{I(t)(\delta_t), S(t)(\delta_t)\} \leq \max\{I(t)(\delta), S(t)(\delta)\}.$$

In order to prove (4) using (1) it is enough to show that $\delta_s < \delta_t$ implies $\Phi(t) < \Phi(s)$. If $\delta_s < \delta_t$ then by the definition of δ_t, the fact that the sequences $I(t)$ and $S(t)$ are strictly decreasing and (3.4.3), we obtain

$$\Phi(t) = \max\{I(t)(\delta_t), S(t)(\delta_t)\} < \max\{I(t)(\delta_s), S(t)(\delta_s)\} \leq \Phi(s),$$

which proves (4).

Now we prove (5). Notice that $t^\frown r \leq_{altlex} t^\frown q$ implies that $J_{t^\frown r} \leq_{altlex\times 2} J_{t^\frown q}$. Thus,

$$\inf(J_{t^\frown r}) \leq_{L\times 2} \sup(J_{t^\frown r}) \leq_{L\times 2} \inf(J_{t^\frown q}) \leq_{L\times 2} \sup(J_{t^\frown q}).$$

Consequently, by the fact that Ψ is order preserving, we get

$$I(t^\frown r) \leq_{altlex} S(t^\frown r) \leq_{altlex} I(t^\frown q) \leq_{altlex} S(t^\frown q). \tag{3.4.4}$$

From $J_{t^\frown r}, J_{t^\frown q} \subset J_t$ it is clear that

$$I(t) \leq_{altlex} I(t^\frown r) \leq_{altlex} S(t^\frown r)$$

$$\leq_{altlex} I(t^\frown q) \leq_{altlex} S(t^\frown q) \leq_{altlex} S(t).$$

Thus, from the definition of δ_t we have

$$I(t)|_{\delta_t} = I(t^\frown r)|_{\delta_t} = S(t^\frown r)|_{\delta_t} = I(t^\frown q)|_{\delta_t} = S(t^\frown q)|_{\delta_t} = S(t)|_{\delta_t},$$

so this shows that (a) holds. Now using (a), the definition of $<_{altlex}$ and (3.4.4) we obtain (b) and (c) of (5) as well.

The proof of (6) is similar to the previous argument: $t \leq_{altlex} u$ implies $J_t \leq_{L\times 2} J_u$, consequently $I(t) \leq_{altlex} S(t) \leq_{altlex} I(u)$. Since by assumption δ is even and $I(t)|_\delta = S(t)|_\delta = I(u)|_\delta$, the definition of $<_{altlex}$ implies

$$I(t)(\delta) \leq S(t)(\delta) \leq I(u)(\delta).$$

□

The following lemma is the essence of our proof.

Lemma 3.4.15. *There exists a $\subsetneq$-increasing sequence $\{s_\alpha\}_{\alpha<\omega_1}$ such that $s_\alpha \in \sigma^*[0,1]$, $l(s_\alpha) = \alpha$ and*

$$(\forall r \in s_\alpha)(\Phi(s_\alpha) < r). \tag{*}$$

Proof. We define s_α by induction on α.

Suppose that we have defined s_β for $\beta < \alpha$. Then by the inductive hypothesis for every $\beta < \alpha$ we have

$$(\forall r \in s_\beta)(\Phi(s_\beta) < r). \tag{3.4.5}$$

Now we define s_α for limit and successor α's separately.

α IS A LIMIT. Let $s_\alpha = \bigcup_{\beta<\alpha} s_\beta$. If $r \in s_\alpha$ is arbitrary then $r \in s_\beta$ for some $\beta < \alpha$. Notice that part (a) of (2) of Lemma 3.4.14 and (3.4.5) imply

$$(s_\beta \subset s_\alpha \text{ and } r \in s_\beta) \Rightarrow \Phi(s_\alpha) \leq \Phi(s_\beta) < r.$$

Hence, using $s_\beta \subset s_\alpha$ we obtain $\Phi(s_\alpha) < r$ so s_α satisfies requirement (*).

α IS A SUCCESSOR. Let $\alpha = \beta + 1$.

Our aim is to find a real x such that

$$s_\beta {}^\frown x \in \sigma^*[0,1] \text{ and } \Phi(s_\beta {}^\frown x) < x. \tag{3.4.6}$$

Clearly, this ensures that $s_\alpha = s_\beta {}^\frown x$ satisfies (*).

Notice that (3.4.5) yields

$$s_\beta {}^\frown \Phi(s_\beta) \in \sigma^*[0,1]. \tag{3.4.7}$$

Now we have to separate two cases.

First, suppose that

$$\Phi(s_\beta {}^\frown \Phi(s_\beta)) < \Phi(s_\beta).$$

Let $x = \Phi(s_\beta)$. It is clear that x satisfies (3.4.6) by induction, so $s_\alpha = s_\beta {}^\frown x$ is a suitable choice for (*).

Second, suppose that $\Phi(s_\beta {}^\frown \Phi(s_\beta)) \geq \Phi(s_\beta)$. Since $s_\beta \subset s_\beta {}^\frown \Phi(s_\beta)$, by part (a) of (2) of Lemma 3.4.14 we have $\Phi(s_\beta {}^\frown \Phi(s_\beta)) \leq \Phi(s_\beta)$, so in fact

$$\Phi(s_\beta {}^\frown \Phi(s_\beta)) = \Phi(s_\beta). \tag{3.4.8}$$

Moreover, by (4) of Lemma 3.4.14 we obtain that (3.4.8) implies

$$\delta_{s_\beta{}^\frown\Phi(s_\beta)} = \delta_{s_\beta}. \tag{3.4.9}$$

In order to find an x that satisfies (3.4.6) we will distinguish 3 cases according to the parity of β and δ_{s_β}.

Case 1. β and δ_{s_β} have the same parity.
By (3.4.2) we can choose an

$$x \in (\phi(s_\beta{}^\frown \Phi(s_\beta)), \Phi(s_\beta{}^\frown \Phi(s_\beta))) = (\phi(s_\beta{}^\frown \Phi(s_\beta)), \Phi(s_\beta)) \tag{3.4.10}$$

where the equality holds because of (3.4.8).

We claim that x has property (3.4.6). Clearly, $x < \Phi(s_\beta)$ and therefore by (3.4.5) we have $s_\beta{}^\frown x \in \sigma^*[0,1]$, hence the first part of (3.4.6) holds. Now we can use (5) of Lemma 3.4.14 (part (b) with $t = s_\beta$, $r = x$, $q = \Phi(s_\beta)$ if δ_{s_β} and β are even and part (c) with $t = s_\beta$, $r = \Phi(s_\beta)$, $q = x$ if they are odd) and we obtain

$$\max\{I(s_\beta{}^\frown x)(\delta_{s_\beta}), S(s_\beta{}^\frown x)(\delta_{s_\beta})\} \leq$$

$$\min\{I(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta}), S(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta})\} = \phi(s_\beta{}^\frown \Phi(s_\beta)) < x, \tag{3.4.11}$$

where the equality follows from the definition of ϕ and (3.4.9) and the last inequality follows from (3.4.10).

By (1) of Lemma 3.4.14 we have $\delta_{s_\beta} \leq \delta_{s_\beta{}^\frown x}$ and (3) of Lemma 3.4.14 implies

$$\Phi(s_\beta{}^\frown x) \leq \max\{I(s_\beta{}^\frown x)(\delta_{s_\beta}), S(s_\beta{}^\frown x)(\delta_{s_\beta})\}.$$

Combining this inequality with (3.4.11) we obtain that the second part of (3.4.6) holds for x. So $s_\alpha = s_\beta{}^\frown x$ satisfies (*), hence we are done with the first case.

Case 2. β is even and δ_{s_β} is odd.
Then clearly, by (3.4.8), (3.4.9) and the odd parity of δ_{s_β}

$$\Phi(s_\beta) = \Phi(s_\beta{}^\frown \Phi(s_\beta)) =$$

$$\max\{I(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta{}^\frown\Phi(s_\beta)}), S(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta{}^\frown\Phi(s_\beta)})\} =$$

$$\max\{I(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta}), S(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta})\} = I(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta}).$$

Thus,

$$\Phi(s_\beta) = I(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta}). \tag{3.4.12}$$

Let $z < \Phi(s_\beta)$ be arbitrary. Clearly, by the parity of β we get $s_\beta{}^\frown z <_{altlex} s_\beta{}^\frown \Phi(s_\beta)$. Hence, using part (c) of (5) of Lemma 3.4.14 with $t = s_\beta$, $r = z$ and $q = \Phi(s_\beta)$ we obtain

$$I(s_\beta{}^\frown z)(\delta_{s_\beta}) \geq S(s_\beta{}^\frown z))(\delta_{s_\beta}) \geq I(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta}) \geq S(s_\beta{}^\frown \Phi(s_\beta))(\delta_{s_\beta}). \tag{3.4.13}$$

Now, part (b) of (2) of Lemma 3.4.14 applied to s_β and $s_\beta {}^\frown z$ yields

$$\max\{I(s_\beta {}^\frown z)(\delta_{s_\beta}), S(s_\beta {}^\frown z)(\delta_{s_\beta})\} \leq \Phi(s_\beta). \tag{3.4.14}$$

Comparing this inequality with (3.4.13) and (3.4.12) we have

$$I(s_\beta {}^\frown z)(\delta_{s_\beta}) = S(s_\beta {}^\frown z)(\delta_{s_\beta}) = I(s_\beta {}^\frown \Phi(s_\beta))(\delta_{s_\beta}). \tag{3.4.15}$$

Therefore, as by (1) of Lemma 3.4.14 $\delta_{s_\beta {}^\frown z} \geq \delta_{s_\beta}$, we obtain that

$$\text{for every } z < \Phi(s_\beta) \text{ we have } \delta_{s_\beta {}^\frown z} \geq \delta_{s_\beta} + 1. \tag{3.4.16}$$

Notice that (a) of (5) of Lemma 3.4.14 applied to $s_\beta {}^\frown z$ and $s_\beta {}^\frown \Phi(s_\beta)$ and (3.4.9) imply that

$$I(s_\beta {}^\frown z)|_{\delta_{s_\beta}} = S(s_\beta {}^\frown z)|_{\delta_{s_\beta}} = I(s_\beta {}^\frown \Phi(s_\beta))|_{\delta_{s_\beta}} = S(s_\beta {}^\frown \Phi(s_\beta))|_{\delta_{s_\beta}}. \tag{3.4.17}$$

Now the even parity of $\delta_{s_\beta} + 1$, $s_\beta {}^\frown z <_{altlex} s_\beta {}^\frown \Phi(s_\beta)$, (3.4.15) and (3.4.17) show that (6) of Lemma 3.4.14 can be applied for $t = s_\beta {}^\frown z$ and $u = s_\beta {}^\frown \Phi(s_\beta)$ and $\delta = \delta_{s_\beta} + 1$. This yields for every $z < \Phi(s_\beta)$ that

$$\begin{aligned} &\max\{I(s_\beta {}^\frown z)(\delta_{s_\beta} + 1), S(s_\beta {}^\frown z)(\delta_{s_\beta} + 1)\} \leq \\ &\leq I(s_\beta {}^\frown \Phi(s_\beta))(\delta_{s_\beta} + 1) < I(s_\beta {}^\frown \Phi(s_\beta))(\delta_{s_\beta}) = \Phi(s_\beta), \end{aligned} \tag{3.4.18}$$

where the last inequality follows from the fact that $I(s_\beta {}^\frown \Phi(s_\beta))$ is strictly decreasing and the equality comes from (3.4.12). So by equations (3.4.16), (3.4.18) and (3) of Lemma 3.4.14 for an $x \in (I(s_\beta {}^\frown \Phi(s_\beta))(\delta_{s_\beta} + 1), \Phi(s_\beta))$ we obtain

$$\begin{aligned} \Phi(s_\beta {}^\frown x) &\leq \max\{I(s_\beta {}^\frown x)(\delta_{s_\beta} + 1), S(s_\beta {}^\frown x)(\delta_{s_\beta} + 1)\} \\ &\leq I(s_\beta {}^\frown \Phi(s_\beta))(\delta_{s_\beta} + 1) < x. \end{aligned}$$

Thus, the second part of (3.4.6) holds for x. The first part is clear from $x < \Phi(s_\beta)$ and (3.4.5), hence $s_\alpha = s_\beta {}^\frown x$ is an appropriate choice for (*).

Case 3. β is odd and δ_{s_β} is even.
Then s_β has a least element $\min s_\beta$, and by induction and (3.4.8) $\min s_\beta > \Phi(s_\beta) = \Phi(s_\beta {}^\frown \Phi(s_\beta))$. Now let $x \in (\Phi(s_\beta), \min s_\beta)$. Then we have $s_\beta {}^\frown x \in \sigma^*[0,1]$, so the first part of (3.4.6) holds. Since β is odd, we have $s_\beta {}^\frown x <_{altlex} s_\beta {}^\frown \Phi(s_\beta)$. Therefore, from the fact that δ_{s_β} is even using part (b) of (5) of Lemma 3.4.14 it follows that

$$\begin{aligned} I(s_\beta {}^\frown x)(\delta_{s_\beta}) &\leq S(s_\beta {}^\frown x)(\delta_{s_\beta}) \leq S(s_\beta {}^\frown \Phi(s_\beta))(\delta_{s_\beta}) \\ &\leq \Phi(s_\beta {}^\frown \Phi(s_\beta)) = \Phi(s_\beta) < x, \end{aligned} \tag{3.4.19}$$

where the last $\leq$ uses (3.4.9) while the equality comes from (3.4.8). Hence, using (1) of Lemma 3.4.14 we get $\delta_{s_\beta {}^\frown x} \geq \delta_{s_\beta}$, so by (3) of Lemma 3.4.14 and (3.4.19) we obtain

$$\Phi(s_\beta {}^\frown x) \leq \max\{I(s_\beta {}^\frown x)(\delta_{s_\beta}), S(s_\beta {}^\frown x)(\delta_{s_\beta})\} < x,$$

thus, again x satisfies the second part of (3.4.6) so $s_\alpha = s_\beta {}^\frown x$ is a good choice for (*).

Thus, in any case we can carry out the induction. □

In order to prove the theorem just notice that Lemma 3.4.15 gives an ω_1-long $\subsetneq$-increasing sequence of elements in $\sigma^*[0,1]$. But then $\bigcup_{\alpha<\omega_1} s_\alpha$ would be an ω_1-long decreasing sequence of reals, which is a contradiction. Therefore no completion of $([0,1]^{<\omega_1}_{\searrow 0}, <_{altlex})$ can be embedded into itself. This finishes the proof of the theorem. □

Remark 3.4.16. Let C be the following set:

$$\{\bar{x}^\frown x_\xi{}^\frown 0 : \bar{x} \in \sigma^*[0,1], \xi \text{ is even}, l(\bar{x}) = \xi + 1, x_\xi \neq 0\}.$$

The ordering $<_{altlex}$ extends to the set $C \cup [0,1]^{<\omega_1}_{\searrow 0}$ naturally and it is not hard to show that this ordering is complete. By Theorem 3.4.12 this is not representable in $\mathcal{B}_1$. However, one can show that this ordering does not contain ω_1, ω_1^* and Suslin lines. Thus, we obtain another proof of [23, Theorem 4.1].

3.5 Proof of Proposition 3.2.5

Proposition 3.2.5. ([42]) Let X be a Polish space and $f \in b\mathcal{B}_1^+(X)$. Then $\Phi(f)$ is defined, $\Phi(f) \in \sigma^* bUSC^+$ and we have

(1) $f = \sum^*_{\beta<\alpha}(-1)^\beta f_\beta + (-1)^\alpha g_\alpha$ for every $\alpha \leq \xi$,

(2) $f_\xi \equiv 0$,

(3) $f = \sum^*_{\alpha<\xi}(-1)^\alpha f_\alpha$.

Proof. First we show that $\Phi(f)$ is defined and $\Phi(f) \in \sigma^* bUSC^+$. In order to prove this, we will show the following lemma.

Lemma 3.5.1. *The functions g_α and f_α (assigned to f in Definition 3.2.4) are bounded nonnegative and the sequence (f_α) is decreasing.*

Proof. It follows trivially from the definition of the upper regularisation that if g is an arbitrary function then

$$g \text{ is bounded} \Rightarrow \widehat{g} \text{ exists, bounded and } \widehat{g} \geq_p g. \tag{3.5.1}$$

Now we prove the statement of the lemma by induction on α. If $\alpha = 0$ then $g_0 = f$ and $f_0 = \widehat{f}$, hence from $f \in b\mathcal{B}_1^+(X)$ and (3.5.1) clearly follows that g_0 and f_0 are bounded nonnegative functions.

If α is a successor then by definition $g_\alpha = \widehat{g_{\alpha-1}} - g_{\alpha-1}$ so by the second part of (3.5.1) we have $g_\alpha \geq_p 0$. Moreover, since $g_{\alpha-1}$ is bounded $\widehat{g_{\alpha-1}}$ is also bounded. Thus, g_α is the difference of two bounded functions, therefore it is also bounded. Therefore, by (3.5.1) f_α exists (notice that we have defined the upper regularisation only for bounded functions) and also bounded and nonnegative.

Now we show that the sequence (f_α) is decreasing. By the nonnegativity of $g_{\alpha-1}$ we have $f_{\alpha-1} - g_{\alpha-1} \leq_p f_{\alpha-1}$, so

$$f_\alpha = \widehat{f_{\alpha-1} - g_{\alpha-1}} \leq_p \widehat{f}_{\alpha-1} = f_{\alpha-1}.$$

For limit α we have

$$g_\alpha = \inf\{g_\beta : \beta < \alpha \text{ and } \beta \text{ is even}\}, \tag{3.5.2}$$

so clearly $g_\alpha \geq_p 0$ and g_α is bounded. Hence using again (3.5.1) we obtain that f_α is bounded and nonnegative.

Now for every β we have $g_\beta \leq_p f_\beta$. Therefore, if β is an even ordinal and $\beta < \alpha$ then by (3.5.2) we have

$$g_\alpha \leq_p g_\beta \leq_p f_\beta,$$

so $f_\alpha = \widehat{g_\alpha} \leq_p \widehat{f_\beta} = f_\beta$. But if β is odd, then $\beta + 1$ is even and $\beta + 1 < \alpha$. Using (3.5.2) we obtain $g_\alpha \leq_p g_{\beta+1}$ hence by the definition of f_α and $f_{\beta+1}$ and the inductive hypothesis we have $f_\alpha \leq_p f_{\beta+1} \leq_p f_\beta$. This finishes the proof of the lemma. □

Clearly, by the definition of upper regularisation, the functions f_α are upper semicontinuous. Therefore, by Lemma 3.5.1 we obtain that (f_α) is a decreasing sequence of nonnegative USC functions, so it must stabilise from some countable ordinal ξ ([44] or Lemma 3.2.7). Therefore, for every function in $f \in b\mathcal{B}_1^+(X)$ we have that $\Phi(f)$ is defined and $\Phi(f) \in \sigma^* bUSC^+(X)$.

Now we need the following lemma.

Lemma 3.5.2. *Let $(f_\alpha)_{\alpha<\xi} \in \sigma^* USC^+$. Then $\sum^*_{\alpha<\xi} (-1)^\alpha f_\alpha$ is a Baire class 1 function.*

Proof. We prove the lemma by induction on ξ.

First, if ξ is a successor just use that Baire class 1 functions are closed under addition and subtraction.

Second, if ξ is a limit, by definition of the alternating sums we have that

$$\textstyle\sum^*_{\alpha<\xi} (-1)^\alpha f_\alpha = \sup\{\sum^*_{\beta<\alpha} (-1)^\beta f_\beta : \alpha < \xi, \alpha \text{ even}\}.$$

For even $\alpha < \xi$ we have

$$\textstyle\sum^*_{\beta<\alpha} (-1)^\beta f_\beta = \sum^*_{\beta<\alpha+1} (-1)^\beta f_\beta - f_\alpha. \tag{*}$$

Again, for even α

$$\textstyle\sum^*_{\beta<\alpha} (-1)^\beta f_\beta + f_\alpha - f_{\alpha+1} = \sum^*_{\beta<\alpha+2} (-1)^\beta f_\beta$$

so since the sequence $(f_\alpha)_{\alpha<\xi}$ is decreasing the sequence $(\sum^*_{\beta<\alpha} (-1)^\beta f_\beta)_{\alpha \text{ even}}$ is increasing. Similarly, the sequence $(\sum^*_{\beta<\alpha+1} (-1)^\beta f_\beta)_{\alpha \text{ even}}$ is decreasing. Notice that if $(r_\beta)_{\beta<\alpha}$ and $(t_\beta)_{\beta<\alpha}$ are decreasing transfinite sequences of nonnegative reals such that $r_\beta - t_\beta$ is increasing, then

$$\sup\{r_\beta - t_\beta : \beta < \alpha\} = \inf\{r_\beta : \beta < \alpha\} - \inf\{t_\beta : \beta < \alpha\}.$$

Therefore, applying (∗) and these facts we have

$$\sup\{\textstyle\sum^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\} =$$

$$\inf\{\textstyle\sum^*_{\beta<\alpha+1}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\} - \inf\{f_\alpha : \alpha < \xi \text{ even}\}.$$

The infimum of USC functions is also USC, hence the right-hand side of the equation is the difference of the infimum of a countable family of Baire class 1 functions and a USC function. Therefore, $\sup\{\sum^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\}$ is the infimum of a countable family of Baire class 1 functions. Moreover, by the inductive hypothesis, this function is also the supremum of a countable family of Baire class 1 functions. Now, using the fact that a function is Baire class 1 if and only if the preimage of every open set is $\mathbf{\Sigma}^0_2(X)$ it is easy to see that if a function h is the infimum of a countable family of Baire class 1 functions then for every $a \in \mathbb{R}$ we have that $h^{-1}((-\infty, a))$ is in $\mathbf{\Sigma}^0_2(X)$. Similarly, if h is the supremum of a countable family of Baire class 1 functions then the sets $h^{-1}((a, \infty))$ are also in $\mathbf{\Sigma}^0_2(X)$. But this implies that a function that is both an infimum and a supremum of countable families of Baire class 1 functions is also Baire class 1.

So, as an infimum and supremum of countable families of Baire class 1 functions, the function $\sup\{\sum^*_{\beta<\alpha}(-1)^\beta f_\beta : \alpha < \xi \text{ even}\}$ is also a Baire class 1 function, which completes the inductive proof. □

Now we prove (1) of the Proposition by induction on α.

For $\alpha = 0$ this is clear. If α is a successor, then $g_{\alpha-1} = f_{\alpha-1} - g_\alpha$, so

$$f = \textstyle\sum^*_{\beta<\alpha-1}(-1)^\beta f_\beta + (-1)^{\alpha-1} g_{\alpha-1} =$$

$$\textstyle\sum^*_{\beta<\alpha-1}(-1)^\beta f_\beta + (-1)^{\alpha-1}(f_{\alpha-1} - g_\alpha) = \sum^*_{\beta<\alpha}(-1)^\beta f_\beta + (-1)^\alpha g_\alpha.$$

For limit α notice that we have by induction for every even $\beta < \alpha$

$$f = \textstyle\sum^*_{\gamma<\beta}(-1)^\gamma f_\gamma + g_\beta.$$

Then, using that the sequence $(f_\beta)_{\beta<\alpha}$ is decreasing, the sequence $(\sum^*_{\gamma<\beta}(-1)^\gamma f_\gamma)_{\beta \text{ even}}$ is increasing, so $(g_\beta)_{\beta \text{ even}}$ is decreasing as their sum is constant f.

Notice that if $(r_\beta)_{\beta<\alpha}$ is an increasing and $(t_\beta)_{\beta<\alpha}$ is a decreasing transfinite sequence of nonnegative reals such that $r_\beta + t_\beta = c$ is constant, then

$$c = \sup\{r_\beta + t_\beta : \beta < \alpha\} = \sup\{r_\beta : \beta < \alpha\} + \inf\{t_\beta : \beta < \alpha\}.$$

So

$$f = \sup_{\beta \text{ even}, \beta<\alpha} \left(\textstyle\sum^*_{\gamma<\beta}(-1)^\gamma f_\gamma + g_\beta\right) =$$

$$\sup_{\beta \text{ even}, \beta<\alpha} \textstyle\sum^*_{\gamma<\beta}(-1)^\gamma f_\gamma + \inf_{\beta \text{ even}, \beta<\alpha} g_\beta = \sum^*_{\beta<\alpha}(-1)^\beta f_\beta + g_\alpha,$$

where the last equality follows from the definition of $\sum^*_{\beta<\alpha}(-1)^\beta f_\beta$ and g_α.

This proves the induction hypothesis, so we have (1).

After rearranging the equality in (1) we have that

$$(-1)^{\alpha+1} g_\alpha = \Sigma^*_{\beta<\alpha} (-1)^\beta f_\beta - f.$$

By Lemma 3.5.2 we have that the sum on the right-hand side of the equation is a Baire class 1 function, therefore g_α is also Baire class 1. We have that $f_{\xi+1} \equiv f_\xi$, so by Definition 3.2.4 we have $\widehat{\widehat{g_\xi} - g_\xi} = \widehat{g_\xi}$. Hence in order to prove (2) it is enough to show the following claim.

Claim. *If g is a nonnegative, bounded Baire class 1 function such that $\widehat{g} = \widehat{\widehat{g} - g}$ then $g \equiv 0$.*

Proof of the Claim. Suppose the contrary. Then there exists an $\varepsilon > 0$ such that $\{x : g(x) > \varepsilon\} \neq \emptyset$. Let $K = \overline{\{x : g(x) > \varepsilon\}}$. Since g is a Baire class 1 function we have that there exists an open set V such that

$$\varepsilon > osc(g, K \cap V) \quad (= \sup_{x,y \in K \cap V} |g(x) - g(y)|)$$

and $K \cap V$ is not empty (see [40, 24.15]).

The function $\limsup_{y \to x} g(y)$ (here in the lim sup we do not exclude those sequences which contain x) is USC. Therefore, by definition $\widehat{g} \leq_p \limsup g$. Hence letting $h = \widehat{g} - g$ we have that

$$h \leq_p \limsup(g) - g. \tag{3.5.3}$$

Now, we claim that

$$(\limsup(g) - g)|_{V \cap K} \leq \varepsilon. \tag{3.5.4}$$

Suppose the contrary. Then there exists an $x \in V \cap K$ such that $(\limsup_{y \to x} g(x)) - g(x) > \varepsilon$. Consequently, there exists a sequence $y_n \to x$, such that $\lim_{n\to\infty} g(y_n) > g(x) + \varepsilon$. Using the nonnegativity of g and the fact that $g|_{K^c} \leq \varepsilon$ we get that $y_n \in K \cap V$ except for finitely many n's. But then $osc(g, K \cap V) > \varepsilon$, a contradiction. So we have (3.5.4) and using (3.5.3) we obtain

$$h|_{V \cap K} \leq \varepsilon. \tag{3.5.5}$$

Observe now that if for a bounded function f and an open set U we have that $f|_U \leq \varepsilon$, then $\widehat{f}|_U \leq \varepsilon$ (clearly, if $|f| < K$ then the function $K \cdot \chi_{U^c} + \varepsilon \cdot \chi_U$ is an USC upper bound of f).

By the above observation used for g on K^c we have that $\widehat{g}|_{K^c} \leq \varepsilon$, in particular from $h = \widehat{g} - g \leq_p \widehat{g}$ we obtain that $h|_{K^c} \leq \varepsilon$. Then from (3.5.5) we get $h|_V \leq \varepsilon$. So finally, using the above observation for h and V we obtain $\widehat{h}|_V \leq \varepsilon$.

The set $\{x : g(x) > \varepsilon\}$ is dense in K, hence there exists an $x_0 \in V \cap \{x : g(x) > \varepsilon\}$. On the one hand $\widehat{g}(x_0) \geq g(x_0) > \varepsilon$, on the other by $x \in V$ we get $\widehat{h}(x_0) \leq \varepsilon$. This contradicts the assumption that $\widehat{g} = \widehat{h}$. □

So we have proved (2) of Proposition 3.2.5.

(3) easily follows from Lemma 3.5.1, (1), (2) since $0 \le g_\xi \le f_\xi \equiv 0$. This finishes the proof of the proposition. □

3.6 Open problems

Probably the most natural and intriguing problem is the following. Recall that the αth level of the Baire hierarchy in a space X is denoted by $\mathcal{B}_\alpha(X)$. Unless stated otherwise, X is an uncountable Polish space.

Problem 3.6.1. *Let $2 \le \alpha < \omega_1$. Characterise the order types of the linearly ordered subsets of $\mathcal{B}_\alpha(X)$. For instance, does there exist a (simple) universal linearly ordered set for $\mathcal{B}_\alpha(X)$? And how about the class of Borel measurable functions $\cup_{\alpha<\omega_1}\mathcal{B}_\alpha(X)$?*

We remark here that Komjáth [43] proved that under the Continuum Hypothesis every ordered set of cardinality at most $\mathfrak{c}$ can be represented in $\mathcal{B}_2(X)$ (hence in $\mathcal{B}_\alpha(X)$ for any $\alpha \ge 2$ as well). Nevertheless, a *ZFC* result would be very interesting and in light of our solution to Laczkovich's problem now it seems conceivable that one can construct relatively simple universal linearly ordered sets in these cases as well. As a first step in this direction it would be interesting to see if the result of Kechris and Louveau can be generalised to $\mathcal{B}_\alpha(X)$. Actually, closely related results from this paper can be generalised from the Baire class 1 case to the Baire class α, as we will see in Chapter 4.

Let $(L_n)_{n\in\omega}$ and L be linearly ordered sets. We say that L is a *blend of* $(L_n)_{n\in\omega}$ if L can be partitioned to pairwise disjoint subsets $(L'_n)_{n\in\omega}$ such that L_n is order isomorphic to L'_n for every n. Elekes [19] proved that if the duplication and completion of every representable ordering was representable then countable blends of representable orderings would also be representable. As we have seen (Theorem 3.4.12), the second condition of this theorem fails, hence it is quite natural to ask the following.

Problem 3.6.2. *Suppose that the linearly ordered sets L_n are representable in $\mathcal{B}_1(X)$ and L is a blend of $(L_n)_{n\in\omega}$. Does it follow that L is also representable in $\mathcal{B}_1(X)$?*

We would expect a negative answer using similar ideas and techniques as in the proof of Theorem 3.4.12.

Elekes and Kunen [21] investigated Problem 3.0.1 in general, for non-Polish X. This raises the next question:

Problem 3.6.3. *Let X be a topological space (e. g. a separable metric space). Characterise the order types of the linearly ordered subsets of $\mathcal{B}_1(X)$. For instance, does there exist a (simple) universal linearly ordered set for $\mathcal{B}_1(X)$?*

We believe that an affirmative answer might be useful in answering Question 3.6.1 using topology refinements.

The next problem concerns characterising all the subposets of our function spaces instead of only the linearly ordered ones. For example, it is not hard to check that $\mathcal{F}(X) = \mathcal{C}([0,1])$ contains an isomorphic copy of a poset P iff $(\mathcal{P}(\omega), \subsetneq)$ does.

Problem 3.6.4. *Characterise, up to poset-isomorphism, the subsets of $\mathcal{B}_1(X)$. Does there exist a simple, informative universal poset? For instance, is $\mathbf{\Delta}^0_2(X)$ or $USC^{<\omega_1}_{\searrow 0}(X)$ universal?*

Here $USC^{<\omega_1}_{\searrow 0}$ is defined analogously to $[0,1]^{<\omega_1}_{\searrow 0}$ and is ordered by the natural modification of $<_{altlex}$. Notice that our method of proving that $(\mathcal{B}_1(X), <_p) \hookrightarrow (\mathbf{\Delta}^0_2(X), \subsetneq)$ does not give a poset isomorphism between $\mathcal{B}_1(X)$ and its image. In fact, the image is linearly ordered. Unfortunately, it can be easily seen that even the Kechris-Louveau-type embedding $\mathcal{B}_1(X) \to bUSC^{<\omega_1}_{\searrow 0}$, that is, assigning to every Baire class 1 function its canonical resolution as a sum is not a poset isomorphism.

At first sight Laczkovich's problem seems to be closely related to the theory of Rosenthal compacta [27].

Problem 3.6.5. *Explore the connection between the topic of our paper and the theory of Rosenthal compacta.*

Chapter 4

Ranks on the Baire class ξ functions

A fundamental tool in the analysis of Baire class 1 functions is the theory of ranks, that is, maps assigning countable ordinals to Baire class 1 functions, typically measuring their complexity. In their seminal paper [42], Kechris and Louveau systematically investigated three very important ranks on the Baire class 1 functions. One can easily see that the theory has no straightforward generalisation to the case of Baire class ξ functions.

Hence the following very natural but somewhat vague question arises.

Question 4.0.1. *Is there a natural extension of the theory of Kechris and Louveau to the case of Baire class ξ functions?*

There is actually a very concrete version of this question that was raised by Elekes and Laczkovich in [22]. In order to be able to formulate this we need some preparation. For $\theta, \theta' < \omega_1$ let us define the relation $\theta \lesssim \theta'$ if $\theta' \leq \omega^\eta \implies \theta \leq \omega^\eta$ for every $1 \leq \eta < \omega_1$ (we use ordinal exponentiation here). Note that $\theta \leq \theta'$ implies $\theta \lesssim \theta'$, while $\theta \lesssim \theta'$, $\theta' > 0$ implies $\theta \leq \theta' \cdot \omega$. We will also use the notation $\theta \approx \theta'$ if $\theta \lesssim \theta'$ and $\theta' \lesssim \theta$. Then $\approx$ is an equivalence relation. Define the translation map $T_t : \mathbb{R} \to \mathbb{R}$ by $T_t(x) = x + t$ for every $x \in \mathbb{R}$.

Question 4.0.2. *([22, Question 6.7]) Is there a map $\rho : \mathcal{B}_\xi(\mathbb{R}) \to \omega_1$ such that*

- *ρ is unbounded in ω_1, moreover, for every non-empty perfect set $P \subseteq \mathbb{R}$ and ordinal $\zeta < \omega_1$ there is a function $f \in \mathcal{B}_\xi(\mathbb{R})$ such that f is 0 outside of P and $\rho(f) \geq \zeta$,*
- *ρ is translation-invariant, i.e., $\rho(f \circ T_t) = \rho(f)$ for every $f \in \mathcal{B}_\xi(\mathbb{R})$ and $t \in \mathbb{R}$,*
- *ρ is essentially linear, i.e., $\rho(cf) \approx \rho(f)$ and $\rho(f+g) \lesssim \max\{\rho(f), \rho(g)\}$ for every $f, g \in \mathcal{B}_\xi(\mathbb{R})$ and $c \in \mathbb{R} \setminus \{0\}$,*
- *$\rho(f \cdot \chi_F) \lesssim \rho(f)$ for every closed set $F \subseteq \mathbb{R}$ and $f \in \mathcal{B}_\xi(\mathbb{R})$?*

The problem is not formulated in this exact form in [22], but a careful examination of the proofs there reveals that this is what they need for their results to go through. Actually, there are numerous equivalent formulations, for example we may simply replace $\lesssim$ by $\leq$ (indeed, just replace ρ satisfying the above properties by $\rho'(f) = \min\{\omega^\eta : \rho(f) \leq \omega^\eta\}$). However, it turns out, as it was already also the case in [42], that $\lesssim$ is more natural here.

The original motivation of Elekes and Laczkovich came from the theory of paradoxical geometric decompositions (like the Banach-Tarski paradox, Tarski's problem of circling the square, etc.). It has turned out that the solvability of certain systems of difference equations plays a key role in this theory.

Definition 4.0.3. Let $\mathbb{R}^\mathbb{R}$ denote the set of functions from $\mathbb{R}$ to $\mathbb{R}$. A *difference operator* is a mapping $D : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}$ of the form

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + b_i),$$

where a_i and b_i are fixed real numbers.

Definition 4.0.4. A *difference equation* is a functional equation

$$Df = g,$$

where D is a difference operator, g is a given function and f is the unknown.

Definition 4.0.5. A *system of difference equations* is

$$D_i f = g_i \quad (i \in I),$$

where I is an arbitrary set of indices.

It is not very hard to show that a system of difference equations is solvable iff every *finite* subsystem is solvable. But if we are interested in continuous solutions then this result is no longer true. However, if every *countable* subsystem of a system has a continuous solution the the whole system has a continuous solution as well. This motivates the following definition, which has turned out to be a very useful tool for finding necessary conditions for the existence of certain solutions.

Definition 4.0.6. Let $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ be a class of real functions. The *solvability cardinal* of $\mathcal{F}$ is the minimal cardinal $sc(\mathcal{F})$ with the property that if every subsystem of size less than $sc(\mathcal{F})$ of a system of difference equations has a solution in $\mathcal{F}$ then the whole system has a solution in $\mathcal{F}$.

It was shown in [22] that the behaviour of $sc(\mathcal{F})$ is rather erratic. For example, $sc(\text{polynomials}) = 3$ but $sc(\text{trigonometric polynomials}) = \omega_1$, $sc(\{f : f \text{ is continuous}\}) = \omega_1$ but $sc(\{f : f \text{is Darboux}\}) = (2^\omega)^+$, and $sc(\mathbb{R}^\mathbb{R}) = \omega$.

It is also proved in their paper that $\omega_2 \leq sc(\{f : f \text{ is Borel}\}) \leq (2^\omega)^+$, therefore if we assume the Continuum Hypothesis then $sc(\{f : f \text{ is Borel}\}) = \omega_2$. Moreover, they obtained that $sc(\mathcal{B}_\xi) \leq (2^\omega)^+$ for every $2 \leq \xi < \omega_1$, and asked if $\omega_2 \leq sc(\mathcal{B}_\xi)$. They noted that a positive answer to Question 4.0.2 would yield a positive answer here.

For more information on the connection between ranks, solvability cardinals, systems of difference equations, liftings, and paradoxical decompositions consult [22], [48], [47] and the references therein.

In order to be able to answer the above questions we need to address one more problem that has already appeared in Chapter 3, where we used another part of the work of Kechris and Louveau. As we have mentioned before, Kechris and Louveau have only worked out their theory in compact metric spaces, while it is really essential for our purposes to be able to apply the results in arbitrary Polish spaces.

Question 4.0.7. *Does the theory of Kechris and Louveau generalise from compact metric spaces to arbitrary Polish spaces?*

Now we describe our results and say a few words about the organisation of the chapter. First we review the results of Kechris and Louveau in quite some detail in Section 4.2, and also answer Question 4.0.7 in the affirmative. Most of the results in this section are not considered to be new, we only have to check that the proofs in [42] work in non-compact Polish spaces as well. A notable exception is Theorem 4.2.35 stating that the three ranks essentially coincide for bounded Baire class 1 functions, since our highly non-trivial proof for the case of general Polish spaces required completely new ideas. Next, in Section 4.3, we propose numerous very natural ranks on the Baire class ξ functions that surprisingly turn out to be bounded in ω_1! Then we answer Question 4.0.1 and Question 4.0.2 in the affirmative in Section 4.4. We actually define four ranks on every $\mathcal{B}_\xi$, but two of these turn out to be essentially equal, and the resulting three ranks are very good analogues of the original ranks of Kechris and Louveau. We are able to generalise most of their results to these new ranks. As a corollary, we also obtain that $\omega_2 \leq sc(\mathcal{B}_\xi)$, and hence if we assume the Continuum Hypothesis then $sc(\mathcal{B}_\xi) = \omega_2$ for every $2 \leq \xi < \omega_1$.

In Section 4.5 we prove that if a rank has certain natural properties then it coincides with α, β and γ on the bounded Baire class 1 functions. We also indicate how one might generalise this to the bounded Baire class ξ case.

Finally, we collect the open questions in Section 4.6.

4.1 Preliminaries

Throughout the chapter, let (X, τ) be an uncountable Polish space.

If τ' is a topology on X then we denote the family of real valued functions defined on X that are of Baire class ξ with respect to τ' by $\mathcal{B}_\xi(\tau')$. In particular, $\mathcal{B}_\xi(X) = \mathcal{B}_\xi(\tau)$. If Y is another Polish space (whose topology is clear from the context) then we also use the notation $\mathcal{B}_\xi(Y)$ for the family of Baire class ξ functions defined on Y. Similarly, $\mathbf{\Sigma}^0_\xi(\tau')$

and $\mathbf{\Sigma}^0_\xi(Y)$ are both the set of $\mathbf{\Sigma}^0_\xi$ subsets, with respect to τ', and in Y, respectively. We use the analogous notations for all the other pointclasses.

If $\mathcal{H}$ is a family of sets then

$$\mathcal{H}_\sigma = \{ \bigcup_{n\in\mathbb{N}} H_n : H_n \in \mathcal{H}\} \text{ and } \mathcal{H}_\delta = \{ \bigcap_{n\in\mathbb{N}} H_n : H_n \in \mathcal{H}\}. \tag{4.1.1}$$

Recall that for $\theta, \theta' < \omega_1$ we write $\theta \lesssim \theta'$ if $\theta' \leq \omega^\eta \implies \theta \leq \omega^\eta$ for every $1 \leq \eta < \omega_1$ (we use ordinal exponentiation here). Note that $\theta \leq \theta'$ implies $\theta \lesssim \theta'$ and $\theta \lesssim \theta'$, $\theta' > 0$ implies $\theta \leq \theta' \cdot \omega$. We write $\theta \approx \theta'$ if $\theta \lesssim \theta'$ and $\theta' \lesssim \theta$. Then $\approx$ is an equivalence relation. For every ordinal θ we have $2\theta < \theta + \omega$, and since ω^η is a limit ordinal for every $\eta \geq 1$ we obtain that $2\theta \approx \theta$ for every ordinal θ.

A rank $\rho : \mathcal{B}_\xi \to \omega_1$ is called *additive* if $\rho(f+g) \leq \max\{\rho(f), \rho(g)\}$ for every $f, g \in \mathcal{B}_\xi$. It is called *linear* if it is additive and $\rho(cf) = \rho(f)$ for every $f \in \mathcal{B}_\xi$ and $c \in \mathbb{R}\setminus\{0\}$. If X is a Polish group then the left and right translation operators are defined as $L_{x_0}(x) = x_0 \cdot x$ ($x \in X$) and $R_{x_0}(x) = x \cdot x_0$ ($x \in X$). A rank $\rho : \mathcal{B}_\xi \to \omega_1$ is called *translation-invariant* if $\rho(f \circ L_{x_0}) = \rho(f \circ R_{x_0}) = \rho(f)$ for every $f \in \mathcal{B}_\xi$ and $x_0 \in X$. We say that it is *essentially additive*, *essentially linear*, and *essentially translation-invariant* if the corresponding inequalities and equations hold with $\lesssim$ and $\approx$. Moreover, ρ is additive, essentially additive etc. *for bounded functions*, if the corresponding relations hold whenever f and g are bounded.

Let $(F_\eta)_{\eta<\lambda}$ be a (not necessarily strictly) decreasing sequence of sets. Let us assume that $F_0 = X$ and that the sequence is *continuous*, that is, $F_\eta = \bigcap_{\theta<\eta} F_\theta$ for every limit η and if λ is a limit then $\bigcap_{\eta<\lambda} F_\eta = \emptyset$. We also use the convention that $F_\eta = \emptyset$ if $\eta \geq \lambda$. We say that a set H is the *transfinite difference of* $(F_\eta)_{\eta<\lambda}$ if $H = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$. It is well-known that a set is in $\mathbf{\Delta}^0_{\xi+1}$ iff it is a transfinite difference of $\mathbf{\Pi}^0_\xi$ sets see e.g. [40, 22.27]. We have to point out here that the monograph [40] does *not* assume that the decreasing sequences are continuous, but when proving that every set in $\mathbf{\Delta}^0_{\xi+1}$ has a representation as a transfinite difference they actually construct continuous sequences, hence this issue causes no difficulty here.

The set of sequences of length k whose terms are elements of the set $\{0, \dots, n-1\}$ is denoted by n^k. For $s \in n^k$ we denote the i-th term of s by $s(i)$. If $l \in \{0, \dots, n-1\}$ then $s^\wedge l$ denotes the sequence in n^{k+1} whose first k terms agree with those of s and whose $k+1$st term is l.

4.2 Ranks on the Baire class 1 functions without compactness

In this section we summarise some results concerning ranks on the Baire class 1 functions, following the work of Kechris and Louveau. We do not consider the results in this section as original, we basically just carefully check that the results of Kechris and Louveau hold without the assumption of compactness of X. This is inevitable, since they assumed compactness throughout their paper but we will need these results in Section 4.4 for arbitrary Polish spaces.

A notable exception is Theorem 4.2.35 stating that the three ranks essentially coincide for bounded Baire class 1 functions. Since our highly non-trivial proof for the case of general Polish spaces required completely new ideas, we consider this result as original in the non-compact case.

The definitions of the ranks will use the notion of a *derivative operation*.

Definition 4.2.1. A *derivative* on the closed subsets of X is a map $D : \mathbf{\Pi}^0_1(X) \to \mathbf{\Pi}^0_1(X)$ such that $D(A) \subseteq A$ and $A \subseteq B \Rightarrow D(A) \subseteq D(B)$ for every $A, B \in \mathbf{\Pi}^0_1(X)$.

Definition 4.2.2. For a derivative D we define the *iterated derivatives* of the closed set F as follows:

$$\begin{aligned} D^0(F) &= F, \\ D^{\eta+1}(F) &= D(D^\eta(F)), \\ D^\eta(F) &= \bigcap_{\theta<\eta} D^\theta(F) \text{ if } \eta \text{ is a limit.} \end{aligned}$$

Definition 4.2.3. Let D be a derivative. The *rank* of D is the smallest ordinal η, such that $D^\eta(X) = \emptyset$, if such ordinal exists, ω_1 otherwise. We denote the rank of D by $\text{rank}(D)$.

Remark 4.2.4. In all our applications D satisfies $D(F) \subsetneq F$ for every non-empty closed set F, and since in a Polish space there is no strictly decreasing sequence of closed sets of length ω_1 (see e.g. [40, 6.9]), the rank of a derivative is always a countable ordinal.

Proposition 4.2.5. *If the derivatives D_1 and D_2 satisfy $D_1(F) \subseteq D_2(F)$ for every closed subset $F \subseteq X$ then* $\text{rank}(D_1) \le \text{rank}(D_2)$.

Proof. It is enough to prove that $D_1^\eta(X) \subseteq D_2^\eta(X)$ for every ordinal η. We prove this by transfinite induction on η. For $\eta = 0$ this is obvious, since $D_1^0(X) = D_2^0(X) = X$. Now suppose this holds for η and we prove it for $\eta + 1$. Since $D_1^\eta(X) \subseteq D_2^\eta(X)$ and D_1 is a derivative, we have $D_1(D_1^\eta(X)) \subseteq D_1(D_2^\eta(X))$. Using this observation and the condition of the proposition for the closed set $D_2^\eta(X)$, we have $D_1^{\eta+1}(X) = D_1(D_1^\eta(X)) \subseteq D_1(D_2^\eta(X)) \subseteq D_2(D_2^\eta(X)) = D_2^{\eta+1}(X)$.

For limit η the claim is an easy consequence of the continuity of the sequences, hence the proof is complete. □

Proposition 4.2.6. *Let $n \ge 1$ and let D, $D_0, \dots, D_{n-1}$ be derivative operations on the closed subsets of X. Suppose that they satisfy the following conditions for arbitrary closed sets F and F':*

$$D(F) \subseteq \bigcup_{k=0}^{n-1} D_k(F), \tag{4.2.1}$$

$$D(F \cup F') \subseteq D(F) \cup D(F'). \tag{4.2.2}$$

Then for these derivatives

$$\operatorname{rank}(D) \lesssim \max_{k<n} \operatorname{rank}(D_k). \tag{4.2.3}$$

Proof. We will prove by induction on η that

$$D^{\omega^\eta}(F) \subseteq \bigcup_{k=0}^{n-1} D_k^{\omega^\eta}(F) \tag{4.2.4}$$

for every closed set F. It is easy to see that proving (4.2.4) is enough, since if η is an ordinal satisfying $\operatorname{rank}(D_k) \leq \omega^\eta$ for every $k < n$ then we have $\operatorname{rank}(D) \leq \omega^\eta$.

Now we prove (4.2.4). The case $\eta = 0$ is exactly (4.2.1). For limit η the statement is obvious, since the sequences are decreasing and continuous. Hence, it remains to prove (4.2.4) for $\eta + 1$ if it holds for η. For this it is enough to show that for every $m \in \omega$

$$D^{\omega^\eta \cdot m \cdot n}(F) \subseteq \bigcup_{k=0}^{n-1} D_k^{\omega^\eta \cdot m}(F), \tag{4.2.5}$$

indeed,

$$D^{\omega^{\eta+1}}(F) = \bigcap_{m\in\omega} D^{\omega^\eta \cdot m \cdot n}(F) \subseteq \bigcap_{m\in\omega} \left(\bigcup_{k=0}^{n-1} D_k^{\omega^\eta \cdot m}(F) \right),$$

hence $x \in D^{\omega^{\eta+1}}(F)$ implies that without loss of generality $x \in D_0^{\omega^\eta \cdot m}(F)$ for infinitely many m, but the sequence $D_0^{\omega^\eta \cdot m}(F)$ is decreasing, hence $x \in \bigcap_{m\in\omega} D_0^{\omega^\eta \cdot m}(F) = D_0^{\omega^{\eta+1}}(F)$.

Now we prove (4.2.5). Let $F_\emptyset = F$, and for $m \in \mathbb{N}$, $s \in n^m$ and $k < n$ let

$$F_{s^\wedge k} = D_k^{\omega^\eta}(F_s).$$

It is enough that for $m \geq 1$

$$D^{\omega^\eta \cdot m}(F) \subseteq \bigcup_{s\in n^m} F_s, \tag{4.2.6}$$

since it is easy to see that

$$\bigcup_{s\in n^{m\cdot n}} F_s \subseteq \bigcup_{k=0}^{n-1} \bigcup \{F_s : s \in n^{m\cdot n} \text{ and } |\{i : s(i) = k\}| \geq m\},$$

yielding (4.2.5), as

$$\bigcup \{F_s : s \in n^{m\cdot n} \text{ and } |\{i : s(i) = k\}| \geq m\} \subseteq D_k^{\omega^\eta \cdot m}(F).$$

It remains to prove (4.2.6) by induction on m. For $m = 1$, this is only the induction hypothesis of (4.2.4) for η. By supposing (4.2.6) for m, we have

$$D^{\omega^\eta \cdot (m+1)}(F) = D^{\omega^\eta}\left(D^{\omega^\eta \cdot m}(F)\right) \subseteq D^{\omega^\eta}\left(\bigcup_{s\in n^m} F_s\right) \subseteq$$

$$\subseteq \bigcup_{s\in n^m} D^{\omega^\eta}(F_s) \subseteq \bigcup_{s\in n^{m+1}} F_s,$$

where we used (4.2.2) ω^η many times for the second containment, and for the last one we used the induction hypothesis, that is (4.2.4) for η. This finishes the proof. □

4.2.1 The separation rank

This rank was first introduced by Bourgain [9].

Definition 4.2.7. Let A and B be two subsets of X. We associate a derivative with them by

$$D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B}. \tag{4.2.7}$$

It is easy to see that $D_{A,B}(F)$ is closed, $D_{A,B}(F) \subseteq F$ and $D_{A,B}(F) \subseteq D_{A,B}(F')$ for every pair of sets A and B and every pair of closed sets $F \subseteq F'$, hence $D_{A,B}$ is a derivative. We use the notation $\alpha(A, B) = \operatorname{rank}(D_{A,B})$.

Definition 4.2.8. The *separation rank* of a Baire class 1 function f is defined as

$$\alpha(f) = \sup_{\substack{p<q\\ p,q\in\mathbb{Q}}} \alpha(\{f \le p\}, \{f \ge q\}). \tag{4.2.8}$$

Remark 4.2.9. Actually,

$$\alpha(f) = \sup_{\substack{x<y\\ x,y\in\mathbb{R}}} \alpha(\{f \le x\}, \{f \ge y\}),$$

since if $x < p < q < y$ then $\alpha(\{f \le x\}, \{f \ge y\}) \le \alpha(\{f \le p\}, \{f \ge q\})$, since any set $H \in \mathbf{\Delta}^0_2(X)$ separating the level sets $\{f \le p\}$ and $\{f \ge q\}$ also separates $\{f \le x\}$ and $\{f \ge y\}$.

Proposition 4.2.10. *If f is a Baire class 1 function then $\alpha(f) < \omega_1$.*

Proof. From the definition of the rank and Remark 4.2.4 it is enough to prove that for any pair of rational numbers $p < q$ and non-empty closed set $F \subseteq X$, $D_{A,B}(F) \subsetneq F$, where $A = \{f \le p\}$ and $B = \{f \ge q\}$. Since f is of Baire class 1, it has a point of continuity restricted to F, hence A and B cannot be both dense in F. Consequently, $D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B} \subsetneq F$, proving the proposition. □

Next we prove that $\alpha(A, B) < \omega_1$ iff A and B can be separated by a transfinite difference of closed sets.

Definition 4.2.11. If the sets A and B can be separated by a transfinite difference of closed sets then let $\alpha_1(A, B)$ denote the length of the shortest such sequence, otherwise let $\alpha_1(A, B) = \omega_1$. We define the *modified separation rank* of a Baire class 1 function f as

$$\alpha_1(f) = \sup_{\substack{p<q\\ p,q\in\mathbb{Q}}} \alpha_1(\{f \le p\}, \{f \ge q\}). \tag{4.2.9}$$

Proposition 4.2.12. *Let A and B two subsets of X. Then*

$$\alpha(A,B) \leq \alpha_1(A,B) \leq 2\alpha(A,B), \text{ hence } \alpha(A,B) \approx \alpha_1(A,B).$$

Proof. For the first inequality we can assume that $\alpha_1(A,B) < \omega_1$, so A and B can be separated by a transfinite difference of closed sets. Let $(F_\eta)_{\eta<\lambda}$ be such a sequence, where $\lambda = \alpha_1(A,B)$. Now we have

$$A \subseteq \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}) \subseteq B^c.$$

It is enough to prove that $D^\eta_{A,B}(X) \subseteq F_\eta$ for every η. We prove this by induction. For $\eta = 0$ this is obvious, since $D^0_{A,B}(X) = F_0 = X$.

Now suppose that $D^\eta_{A,B}(X) \subseteq F_\eta$. We show that $D^{\eta+1}_{A,B}(X) = \overline{D^\eta_{A,B}(X) \cap A} \cap \overline{D^\eta_{A,B}(X) \cap B} \subseteq F_{\eta+1}$. If η is even then

$$D^\eta_{A,B}(X) \setminus F_{\eta+1} \subseteq F_\eta \setminus F_{\eta+1} \subseteq B^c,$$

hence $D^\eta_{A,B}(X) \cap B \subseteq F_{\eta+1}$. Since $F_{\eta+1}$ is closed, we obtain $\overline{D^\eta_{A,B}(X) \cap B} \subseteq F_{\eta+1}$, hence $D^{\eta+1}_{A,B} \subseteq F_{\eta+1}$. If η is odd then $F_\eta \setminus F_{\eta+1}$ is disjoint from $\bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$, hence $F_\eta \setminus F_{\eta+1} \subseteq A^c$, and an argument analogous to the above one yields $\overline{D^\eta_{A,B}(X) \cap A} \subseteq F_{\eta+1}$, hence $D^{\eta+1}_{A,B} \subseteq F_{\eta+1}$.

If η is limit and $D^\theta_{A,B}(X) \subseteq F_\theta$ for every $\theta < \eta$ then $D^\eta_{A,B}(X) \subseteq F_\eta$ because the sequences $D^\eta_{A,B}(X)$ and F_η are continuous.

For the second inequality we suppose that $\alpha(A,B) < \omega_1$, that is, the sequence $D^\eta_{A,B}(X)$ terminates at the empty set at some countable ordinal. Let

$$F_{2\eta} = D^\eta_{A,B}(X), \quad F_{2\eta+1} = \overline{D^\eta_{A,B}(X) \cap B}.$$

Clearly, $F_0 = X$ and $F_{2\eta} \supseteq F_{2\eta+1}$ for every η. It is easily seen from the definition of $D^{\eta+1}_{A,B}(X)$ that $F_{2\eta+1} \supseteq F_{2\eta+2}$ for every η. Moreover, the sequence $F_{2\eta} = D^\eta_{A,B}(X)$ is continuous. This implies that the sequence formed by the F_η's is decreasing and continuous.

Now we show that the transfinite difference of this sequence separates A and B.

Every ring of the form $F_{2\eta} \setminus F_{2\eta+1}$ is disjoint from B, so we only need to prove that A is contained in the union of these rings. We show that A is disjoint from the complement of this union by proving that

$$(F_{2\eta+1} \setminus F_{2\eta+2}) \cap A = \left(\overline{D^\eta_{A,B}(X) \cap B} \setminus D^{\eta+1}_{A,B}(X)\right) \cap A = \emptyset$$

for every η. From the definition of the derivative, $D^{\eta+1}_{A,B}(X) = \overline{D^\eta_{A,B}(X) \cap A} \cap \overline{D^\eta_{A,B}(X) \cap B}$. Using that $D^\eta_{A,B}(X)$ is closed, for a point $x \in A \cap \overline{D^\eta_{A,B}(X) \cap B}$ we have $x \in \overline{D^\eta_{A,B}(X) \cap A}$, hence $x \in D^{\eta+1}_{A,B}(X)$. □

Remark 4.2.13. It is claimed in [42] that if X is compact and $\alpha(A,B)=\lambda+n$ with λ limit and $0<n\in\omega$ then $\alpha_1(A,B)$ is either $\lambda+2n$ or $\lambda+2n-1$. However, this does not seem to be true. For a counterexample, let X be the $2n+1$-dimensional cube in $\mathbb{R}^{2n+1}$. Let $A=(F_0\setminus F_1)\cup(F_2\setminus F_3)\cup\dots\cup(F_{2n}\setminus F_{2n+1})$, where F_i is a $(2n+1-i)$-dimensional face of X, and $F_{i+1}\subseteq F_i$ for $i\le 2n$. Let $B=X\setminus A$. The definition of A shows that $\alpha_1(A,B)\le 2n+2$.

Now $D^0_{A,B}(X)=X=F_0$, and by induction, $D^i_{A,B}(X)=F_i$ for $0\le i\le 2n+1$, since $D^i_{A,B}(X)=D(D^{i-1}_{A,B}(X))=D_{A,B}(F_{i-1})=\overline{F_{i-1}\cap A}\cap\overline{F_{i-1}\cap B}=F_i$. Now we have $D^{2n+2}_{A,B}(X)=D_{A,B}(D^{2n+1}_{A,B}(X))=D_{A,B}(F_{2n+1})=\emptyset$, proving that in this case $\alpha(A,B)=2n+2$. Using Proposition 4.2.12 this shows that $\alpha_1(A,B)=\alpha(A,B)=2n+2$.

We leave the proof of the following corollary to the reader.

Corollary 4.2.14. *If f is a Baire class 1 function then*

$$\alpha(f)\le\alpha_1(f)\le 2\alpha(f),\ \textit{hence}\ \alpha(f)\approx\alpha_1(f).$$

Corollary 4.2.15. *If f is a Baire class 1 function then $\alpha_1(f)<\omega_1$.*

Proof. It is an easy consequence of the previous corollary and Proposition 4.2.10. □

4.2.2 The oscillation rank

This rank was investigated by numerous authors, see e.g. [36].

First, we define the oscillation of a function, then turn to the oscillation rank.

Definition 4.2.16. The *oscillation* of a function $f:X\to\mathbb{R}$ at a point $x\in X$ restricted to a closed set $F\subseteq X$ is

$$\omega(f,x,F)=\inf\left\{\sup_{x_1,x_2\in U\cap F}|f(x_1)-f(x_2)| : U\text{ open},\ x\in U\right\}. \tag{4.2.10}$$

Definition 4.2.17. For each $\varepsilon>0$ consider the derivative defined by

$$D_{f,\varepsilon}(F)=\{x\in F:\omega(f,x,F)\ge\varepsilon\}. \tag{4.2.11}$$

It is obvious that $D_{f,\varepsilon}(F)$ is closed, $D_{f,\varepsilon}(F)\subseteq F$ and $D_{f,\varepsilon}(F)\subseteq D_{f,\varepsilon}(F')$ for every function $f:X\to\mathbb{R}$, every $\varepsilon>0$ and every pair of closed sets $F\subseteq F'$, hence $D_{f,\varepsilon}$ is a derivative. Let us denote the rank of $D_{f,\varepsilon}$ by $\beta(f,\varepsilon)$.

Definition 4.2.18. The *oscillation rank* of a function f is

$$\beta(f)=\sup_{\varepsilon>0}\beta(f,\varepsilon). \tag{4.2.12}$$

Proposition 4.2.19. *If f is a Baire class 1 function then $\beta(f)<\omega_1$.*

Proof. Using Remark 4.2.4, it is enough to prove $D_{f,\varepsilon}(F)\subsetneq F$ for every $\varepsilon>0$ and every non-empty closed set $F\subseteq X$. And this is easy, since f restricted to F is continuous at a point $x\in F$, and thus $x\notin D_{f,\varepsilon}(F)$, hence $D_{f,\varepsilon}(F)\subsetneq F$. □

4.2.3 The convergence rank

Now we turn to the convergence rank following Zalcwasser [62] and Gillespie and Hurwitz [32].

Definition 4.2.20. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real valued continuous functions on X. The *oscillation* of this sequence at a point x restricted to a closed set $F \subseteq X$ is

$$\omega((f_n)_{n\in\mathbb{N}}, x, F) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup \{|f_m(y) - f_n(y)| : n, m \geq N,\ y \in U \cap F\}. \quad (4.2.13)$$

Definition 4.2.21. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of real valued continuous functions, and for each $\varepsilon > 0$, define a derivative as

$$D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \{x \in F : \omega((f_n)_{n\in\mathbb{N}}, x, F) \geq \varepsilon\}. \quad (4.2.14)$$

It is easy to see that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F)$ is closed, $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subseteq F$ and $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subseteq D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F')$ for every sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$, every $\varepsilon > 0$ and every pair of closed sets $F \subseteq F'$, hence $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ is a derivative. Let us denote the rank of $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ by $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon)$.

Definition 4.2.22. For a Baire class 1 function f let the *convergence rank* of f be defined by

$$\gamma(f) = \min \left\{ \sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) : \forall n\ f_n \text{ is continuous and } f_n \to f \text{ pointwise} \right\}. \quad (4.2.15)$$

Proposition 4.2.23. *If f is a Baire class 1 function then $\gamma(f) < \omega_1$.*

Proof. It suffices to show that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) \subsetneq F$ for every $\varepsilon > 0$, every non-empty closed set $F \subseteq X$ and every sequence of pointwise convergent continuous functions $(f_n)_{n\in\mathbb{N}}$. Suppose the contrary, then for every N the set $G_N = \{x \in F : \exists n, m \geq N\ |f_n(x) - f_m(x)| > \frac{\varepsilon}{2}\}$ is dense in F. It is also open in F, hence by the Baire category theorem there is a point $x \in F$ such that $x \in G_N$ for every $N \in \mathbb{N}$, hence the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge at x, contradicting our assumption. □

4.2.4 Properties of the ranks

Theorem 4.2.24. *If f is a Baire class 1 function then $\alpha(f) \leq \beta(f) \leq \gamma(f)$.*

Proof. For the first inequality, it is enough to prove that for every $p, q \in \mathbb{Q}$, $p < q$ we can find $\varepsilon > 0$ such that $\alpha(\{f \leq p\}, \{f \geq q\}) \leq \beta(f, \varepsilon)$. Let $A = \{f \leq p\}$, $B = \{f \geq q\}$ and $\varepsilon = p - q$. Using Proposition 4.2.5 it suffices to show that $D_{A,B}(F) \subseteq D_{f,\varepsilon}(F)$ for every $F \in \mathbf{\Pi}^0_1(X)$. If $x \in F \setminus D_{f,\varepsilon}(F)$ then x has a neighbourhood U such that $\sup_{x_1,x_2\in U\cap F} |f(x_1) - f(x_2)| < \varepsilon = p - q$, hence U cannot intersect both A and B. So $x \notin D_{A,B}(F)$, proving the first inequality.

For the second inequality, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to a function f. It is enough to show that $\beta(f,\varepsilon) \leq \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon/3)$. Similarly to the first paragraph we show that $D_{f,\varepsilon}(F) \subseteq D_{(f_n)_{n\in\mathbb{N}},\varepsilon/3}(F)$ for every $F \in \mathbf{\Pi}^0_1(X)$. It is enough to show that if $x \in F \setminus D_{(f_n)_{n\in\mathbb{N}},\varepsilon/3}(F)$ then $x \notin D_{f,\varepsilon}(F)$. For such an x there is a neighbourhood U of x and an $N \in \mathbb{N}$ such that for all $n, m \geq N$ and $x' \in F \cap U$, $|f_n(x') - f_m(x')| < \varepsilon/3$. Letting $m \to \infty$ we get $|f_n(x') - f(x')| \leq \varepsilon/3$ for all $n \geq N$ and $x' \in F \cap U$. Let $V \subseteq U$ be a neighbourhood of x for which $\sup_V f_N - \inf_V f_N < \varepsilon/6$. Now for every $x', x'' \in V \cap F$ we have

$$|f(x') - f(x'')| \leq |f_N(x') - f_N(x'')| + 2\frac{\varepsilon}{3} < \frac{5}{6}\varepsilon < \varepsilon,$$

showing that $x \notin D_{f,\varepsilon}(F)$. □

Proposition 4.2.25. *If X is a Polish group then the ranks α, β and γ are translation invariant.*

Proof. Note first that for a Baire class 1 function f and $x_0 \in X$ the functions $f \circ L_{x_0}$ and $f \circ R_{x_0}$ are also of Baire class 1. Since the topology of a topological group is translation invariant, and the the definitions of the ranks depend only on the topology of the space, the proposition easily follows. □

Theorem 4.2.26. *The ranks are unbounded in ω_1, actually unbounded already on the characteristic functions.*

We postpone the proof, since later we will prove the more general Theorem 4.3.3.

Proposition 4.2.27. *If f is continuous then $\alpha(f) = \beta(f) = \gamma(f) = 1$.*

Proof. In order to prove $\alpha(f) = 1$, consider the derivative $D_{\{f\leq p\},\{f\geq q\}}$, where $p < q$ is a pair of rational numbers. Since the level sets $\{f \leq p\}$ and $\{f \geq q\}$ are disjoint closed sets, $D_{\{f\leq p\},\{f\geq q\}}(X) = \emptyset$.

For $\beta(f) = 1$, note that a continuous function f has oscillation 0 at every point restricted to every set, hence $D_{f,\varepsilon}(X) = \emptyset$ for every $\varepsilon > 0$.

And finally for $\gamma(f) = 1$ consider the sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$, for which $f_n = f$ for every $n \in \mathbb{N}$. It is easy to see that $\omega((f_n)_{n\in\mathbb{N}}, x, F) = 0$ for every point $x \in X$ and every closed set $F \subseteq X$. Now we have that $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) = \emptyset$ for every $\varepsilon > 0$, hence $\gamma(f) = 1$. □

Theorem 4.2.28. *If f is a Baire class 1 function and $F \subseteq X$ is closed then $\alpha(f \cdot \chi_F) \leq 1 + \alpha(f)$, $\beta(f \cdot \chi_F) \leq 1 + \beta(f)$ and $\gamma(f \cdot \chi_F) \leq 1 + \gamma(f)$.*

Proof. First we prove the statement for the ranks α and β. Let D be a derivative either of the form $D_{A,B}$ or of the form $D_{f,\varepsilon}$ where $A = \{f \leq p\}$ and $B = \{f \geq q\}$ for a pair of rational numbers $p < q$ and $\varepsilon > 0$. Let $\overline{D}$ be the corresponding derivative for the function $f \cdot \chi_F$, i.e. $\overline{D} = D_{A',B'}$ or $\overline{D} = D_{f\cdot\chi_F,\varepsilon}$, where $A' = \{f \cdot \chi_F \leq p\}$ and $B' = \{f \cdot \chi_F \geq q\}$.

Since the function $f \cdot \chi_F$ is constant 0 on the open set $X \setminus F$, it is easy to check that in both cases $\overline{D}(X) \subseteq F$. And since the functions f and $f \cdot \chi_F$ agree on F, we have by transfinite induction that $\overline{D}^{1+\eta}(X) \subseteq D^{\eta}(X)$ for every countable ordinal η, implying that $\alpha(f \cdot \chi_F) \leq 1 + \alpha(f)$ and also $\beta(f \cdot \chi_F) \leq 1 + \beta(f)$.

Now we prove the statement for γ. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to f with $\sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) = \gamma(f)$. Let $g_n(x) = 1 - \min\{1, n \cdot d(x, F)\}$ and set $f'_n(x) = f_n(x) \cdot g_n(x)$. It is easy to check that for every n the function f'_n is continuous and $f'_n \to f \cdot \chi_F$ pointwise. For every $x \in X \setminus F$ there is a neighbourhood of x such that for large enough n the function f'_n is 0 on this neighbourhood, hence $D_{(f'_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq F$ for every $\varepsilon > 0$. From this point on the proof is similar to the previous cases, since the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(f'_n)_{n\in\mathbb{N}}$ agree on F, hence, by transfinite induction $D^{1+\eta}_{(f'_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq D^{\eta}_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X)$ for every $\varepsilon > 0$. From this we have $\gamma((f'_n)_{n\in\mathbb{N}}, \varepsilon) \leq 1 + \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon)$ for every $\varepsilon > 0$, hence $\gamma(f \cdot \chi_F) \leq 1 + \gamma(f)$. Thus the proof of the theorem is complete. □

Theorem 4.2.29. *The ranks β and γ are essentially linear.*

Proof. It is easy to see that $\beta(cf) = \beta(f)$ and $\gamma(cf) = \gamma(f)$ for every $c \in \mathbb{R} \setminus \{0\}$, hence it suffices to show that β and γ are essentially additive.

First we consider a modification of the definition of the rank β as follows. Let β_0 be the rank obtained by simply replacing $\sup_{x_1,x_2\in U\cap F} |f(x_1) - f(x_2)|$ in (4.2.10) by $\sup_{x_1\in U\cap F} |f(x) - f(x_1)|$ in the definition of β. Clearly, $\beta_0(f, \varepsilon) \leq \beta(f, \varepsilon) \leq \beta_0(f, \varepsilon/2)$, hence actually $\beta_0 = \beta$. Therefore it is sufficient to prove the theorem for β_0.

To prove the theorem for β_0, let $D_0 = D_{f,\varepsilon/2}$, $D_1 = D_{g,\varepsilon/2}$ and $D = D_{f+g,\varepsilon}$ (we use here the derivatives defining β_0). We show that the conditions of Proposition 4.2.6 hold for these derivatives.

For condition (4.2.1), let $x \in D_{f+g,\varepsilon}(F)$. Since $\omega(f + g, x, F) \geq \varepsilon$, we have $\omega(f, x, F)$ or $\omega(g, x, F) \geq \varepsilon/2$, hence $x \in D_{f,\varepsilon/2}(F) \cup D_{g,\varepsilon/2}(F)$.

Condition (4.2.2) is similar, let $x \in (F \cup F') \setminus (D_{f+g,\varepsilon}(F) \cup D_{f+g,\varepsilon}(F'))$. Since $x \notin D_{f+g,\varepsilon}(F)$, there is a neighbourhood U of x with $|(f + g)(x) - (f + g)(x')| < \varepsilon' < \varepsilon$ for $x' \in U \cap F$. And similarly, there is a neighbourhood U' with $|(f+g)(x)-(f+g)(x')| < \varepsilon'' < \varepsilon$ for $x' \in U' \cap F'$. Now the neighbourhood $U \cap U'$ shows that $\omega(f + g, x, F \cup F') < \varepsilon$, proving that $x \notin D_{f+g,\varepsilon}(F \cup F')$.

The proposition yields that $\beta_0(f+g, \varepsilon) \lesssim \max\{\beta_0(f, \varepsilon/2), \beta_0(g, \varepsilon/2)\}$, hence $\beta_0(f+g) \lesssim \max\{\beta_0(f), \beta_0(g)\}$. This proves the statement for β_0, hence for β.

For γ, we do the same, prove the conditions of the proposition for $D_0 = D_{(f_n)_{n\in\mathbb{N}},\varepsilon/2}$, $D_1 = D_{(g_n)_{n\in\mathbb{N}},\varepsilon/2}$ and $D = D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}$, and use the conclusion of the proposition to finish the proof.

For condition (4.2.1), let $x \in F \setminus \big(D_{(f_n)_{n\in\mathbb{N}},\varepsilon/2}(F) \cup D_{(g_n)_{n\in\mathbb{N}},\varepsilon}(F)\big)$. Now we can choose a common open set $x \in U$ and a common $N \in \mathbb{N}$ such that for all $n, m \geq N$ and

$y \in U \cap F$ we have $|f_n(y) - f_m(y)| \le \varepsilon' < \varepsilon/2$ and $|g_n(y) - g_m(y)| \le \varepsilon' < \varepsilon/2$ (again, with a common $\varepsilon' < \varepsilon/2$). But from this we have $|(f_n + g_n)(y) - (f_m + g_m)(y)| \le 2\varepsilon' < \varepsilon$ for all $n, m \ge N$ and $y \in U \cap F$, so $x \notin D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F)$, yielding (4.2.1).

For (4.2.2) let $x \in (F \cup F') \setminus \big(D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F) \cup D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F')\big)$. For this x we have a neighbourhood U of x, $N \in \mathbb{N}$ and $\varepsilon' < \varepsilon$, such that $|(f_n+g_n)(y) - (f_m+g_m)(y)| \le \varepsilon'$ for every $n, m \ge N$ and $y \in U \cap F$. Similarly, we can find a neighbourhood U', $N' \in \mathbb{N}$ and $\varepsilon'' < \varepsilon$, such that $|(f_n+g_n)(y) - (f_m+g_m)(y)| \le \varepsilon''$ for every $n, m \ge N'$ and $y \in U' \cap F'$. From this, $\omega((f_n+g_n)_{n\in\mathbb{N}}, x, F \cup F') \le \max\{\varepsilon', \varepsilon''\} < \varepsilon$, hence $x \notin D_{(f_n+g_n)_{n\in\mathbb{N}},\varepsilon}(F \cup F')$.

Therefore the proof of the theorem is complete. □

Remark 4.2.30. The analogous result does not hold for the rank α. To see this note first that $\alpha(A, A^c)$ can be arbitrarily large below ω_1 when A ranges over $\mathbf{\Delta}^0_2(X)$. This is a classical fact and we prove a more general result in Corollary 4.3.4.

First we check that for every $A \in \mathbf{\Delta}^0_2(X)$ the characteristic function χ_A can be written as the difference of two upper semicontinuous (usc) functions. Indeed, let $(K_n)_{n\in\omega}$ and $(L_n)_{n\in\omega}$ be increasing sequences of closed sets with $A = \bigcup_n K_n$ and $A^c = \bigcup_n L_n$, and let

$$f_0 = \begin{cases} 0 & \text{on } K_0 \cup L_0, \\ -n & \text{on } (K_n \cup L_n) \setminus (K_{n-1} \cup L_{n-1}) \text{ for } n \ge 1 \end{cases}$$

and

$$f_1 = \begin{cases} 0 & \text{on } L_0, \\ -1 & \text{on } (K_0 \cup L_1) \setminus L_0, \\ -n & \text{on } (K_{n-1} \cup L_n) \setminus (K_{n-2} \cup L_{n-1}) \text{ for } n \ge 2. \end{cases}$$

Then f_0 and f_1 are usc functions with $\chi_A = f_0 - f_1$.

Now we complete the remark by showing that $\alpha(f) \le 2$ for every usc function f. For $p < q$ let $A = \{f \le p\}$ and $B = \{f \ge q\}$. Then B is closed, so $D_{A,B}(X) = \overline{X \cap A} \cap \overline{X \cap B} = \overline{X \cap A} \cap B \subseteq B$. Hence $D^2_{A,B}(X) \subseteq D_{A,B}(B) = \overline{A \cap B} \cap B = \emptyset \cap B = \emptyset$.

Remark 4.2.31. One can easily deduce from Theorem 4.2.29 that $\beta(f \cdot g) \lesssim \max\{\beta(f), \beta(g)\}$ whenever f and g are *bounded* Baire class 1 functions, and similarly for γ. However, we do not know if this holds for arbitrary Baire class 1 functions.

Question 4.2.32. *Are the ranks β and γ* essentially multiplicative *on the Baire class* 1 *functions, that is, does $\beta(f \cdot g) \lesssim \max\{\beta(f), \beta(g)\}$ and $\gamma(f \cdot g) \lesssim \max\{\gamma(f), \gamma(g)\}$ hold whenever f and g are Baire class* 1 *functions?*

Proposition 4.2.33. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\beta(f) \le \sup_n \beta(f_n)$.*

Proof. If $|f - f_n| < \varepsilon/3$ then $|\omega(f, x, F) - \omega(f_n, x, F)| \le \frac{2}{3}\varepsilon$ for every x and F. Therefore $D_{f,\varepsilon}(F) \subseteq D_{f_n,\varepsilon/3}(F)$ for every F, which in turn implies $\beta(f, \varepsilon) \le \beta(f_n, \varepsilon/3)$, from which the proposition easily follows. □

Proposition 4.2.34. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\gamma(f) \lesssim \sup\limits_n \gamma(f_n)$.*

Proof. By taking a subsequence we can suppose that $|f_n(x)-f(x)| \le \frac{1}{2^n}$ for every $n \in \mathbb{N}$ and every $x \in X$. With $g_n(x) = f_n(x) - f_{n-1}(x)$ we have $|g(x)| \le \frac{3}{2^n}$, hence $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent, and $f(x) = f_0(x) + \sum_{n=1}^{\infty} g_n(x)$. Using Theorem 4.2.29 we have $\gamma(g_n) \lesssim \max\{\gamma(f_n), \gamma(f_{n-1})\}$, hence $\sup_n \gamma(g_n) \lesssim \sup_n \gamma(f_n)$. It is enough to prove that for $g = \sum_{n=1}^{\infty} g_n$ we have $\gamma(g) \lesssim \sup_n \gamma(g_n)$, since Theorem 4.2.29 yields $\gamma(f) \lesssim \max\{\gamma(f_0), \gamma(g)\}$.

Now for every $n \in \mathbb{N}$ let $(\varphi_n^k)_{k\in\mathbb{N}}$ be a sequence of continuous functions converging pointwise to g_n with $\sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon) = \gamma(g_n)$. It is easy to see that we can suppose $|\varphi_n^k(x)| \le \frac{3}{2^n}$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$, since by replacing $(\varphi_n^k)_{k\in\mathbb{N}}$ with $\left(\max\left(\min\left(\varphi_n^k, \frac{3}{2^n}\right), -\frac{3}{2^n}\right)\right)_{k\in\mathbb{N}}$ we have a sequence of continuous functions satisfying this, and the sequence is still converging pointwise to g_n, while $\gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon)$ is not increased.

Let $\phi_k = \sum_{n=0}^{k} \varphi_n^k$. We show that $(\phi_k)_{k\in\mathbb{N}}$ converges pointwise to g and also that $\gamma(g) \le \sup_{\varepsilon>0} \gamma((\phi_k)_{k\in\mathbb{N}}, \varepsilon) \lesssim \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon) = \sup_n \gamma(g_n)$, which finishes the proof. To prove pointwise convergence, let $\varepsilon > 0$ be arbitrary and fix $K \in \mathbb{N}$ with $\frac{6}{2^K} < \varepsilon$. For $k > K$ we have

$$|\phi_k(x) - g(x)| = \left|\sum_{n=0}^{k} \varphi_n^k(x) - g(x)\right| \le \left|\sum_{n=0}^{K} \varphi_n^k(x) - g(x)\right| + \left|\sum_{n=K+1}^{k} \varphi_n^k(x)\right|,$$

where the first term of the last expression tends to $\left|\sum_{n=0}^{K} g_n(x) - g(x)\right| \le \frac{3}{2^K}$, while the second is at most $\frac{3}{2^K}$. Hence $\limsup_{k\to\infty} |\phi_k(x) - g(x)| \le 2\frac{3}{2^K} < \varepsilon$ for every $\varepsilon > 0$, showing that $\phi_k(x) \to g(x)$.

Now fix an $\varepsilon > 0$ and $K \in \mathbb{N}$ as before, it is enough to show that $\gamma((\phi_k)_{k\in\mathbb{N}}, 3\varepsilon) \lesssim \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon)$.

For any $x \in X$ and $k, l > K$ we have

$$\begin{aligned} |\phi_k(x) - \phi_l(x)| &= \left|\sum_{n=0}^{k} \varphi_n^k(x) - \sum_{n=0}^{l} \varphi_n^l(x)\right| \\ &\le \sum_{n=0}^{K} \left|\varphi_n^k(x) - \varphi_n^l(x)\right| + \left|\sum_{n=K+1}^{k} \varphi_n^k(x)\right| + \left|\sum_{n=K+1}^{l} \varphi_n^l(x)\right|. \end{aligned} \tag{4.2.16}$$

As before, the sum of the last two terms is at most ε. We want to use Proposition 4.2.6 for the derivatives $D = D_{(\phi_k)_{k\in\mathbb{N}}, 3\varepsilon}$ and $D_n = D_{(\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}}$ for $n \le K$. To check condition (4.2.1), let $x \in F \setminus \bigcup_{n=0}^{K} D_{(\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}}(F)$. Then we have a neighbourhood U of x and an $N \in \mathbb{N}$ such that $\left|\varphi_n^k(y) - \varphi_n^l(y)\right| < \frac{\varepsilon}{K+1}$ for every $n \le K$, every $y \in U \cap F$ and every $k, l \ge N$. This observation and (4.2.4) yields that $|\phi_k(y) - \phi_l(y)| \le 2\varepsilon$ for every $y \in U \cap F$ and $k, l \ge N$ showing that $x \notin D_{(\phi_k)_{k\in\mathbb{N}}, 3\varepsilon}(F)$.

Condition (4.2.2) is similar, and it can be seen as in the proof of Theorem 4.2.29. Now Proposition 4.2.6 gives

$$\gamma((\phi_k)_{k\in\mathbb{N}}, 3\varepsilon) \lesssim \max_{n\le K} \gamma\left((\varphi_n^k)_{k\in\mathbb{N}}, \frac{\varepsilon}{K+1}\right) \le \sup_n \sup_{\varepsilon>0} \gamma((\varphi_n^k)_{k\in\mathbb{N}}, \varepsilon),$$

completing the proof. $\square$

Theorem 4.2.35. *If f is a bounded Baire class 1 function then $\alpha(f) \approx \beta(f) \approx \gamma(f)$.*

Proof. Using Theorem 4.2.24, it is enough to prove that $\gamma(f) \lesssim \alpha(f)$. First, we prove the theorem for characteristic functions.

Lemma 4.2.36. *Suppose that $A \in \mathbf{\Delta}^0_2$. Then $\gamma(\chi_A) \lesssim \alpha(\chi_A)$.*

Proof. In order to prove this, first we have to produce a sequence of continuous functions converging pointwise to χ_A.

For this let $(F_\eta)_{\eta<\lambda}$ be a continuous transfinite decreasing sequence of closed sets, such that

$$A = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1})$$

and $\lambda \approx \alpha(\chi_A)$ given by Corollary 4.2.14. We can assume that the last element of the sequence $(F_\eta)_{\eta<\lambda}$ is $\emptyset$, hence every $x \in X$ is contained in a unique set of the form $F_\eta \setminus F_{\eta+1}$.

For each $k \in \omega$ and $\eta < \lambda$ let $f^k_\eta : X \to [0,1]$ be a continuous function such that $f^k_\eta | F_\eta \equiv 1$, and whenever $x \in X$ and $d(x, F_\eta) \geq \frac{1}{k}$ then $f^k_\eta(x) = 0$. Such a function exists by Urysohn's lemma, since the sets F_η and $\{x \in X : d(x, F_\eta) \geq \frac{1}{k}\}$ are disjoint closed sets.

Now let (η_n) be an enumeration of λ in type $\leq \omega$. Let us define

$$f_k = \sum_{\substack{n\leq k \\ \eta_n \text{ even}}} f^k_{\eta_n} - f^k_{\eta_n+1}.$$

Since the functions f_k are finite sums of continuous functions, they are continuous. We claim that $f_k \to \chi_A$ as $k \to \infty$.

To see this, first let $x \in X$ be arbitrary. Then there exists a unique m such that $x \in F_{\eta_m} \setminus F_{\eta_m+1}$. Choose $k \in \omega$ such that $k \geq m$ and $d(x, F_{\eta_m+1}) \geq \frac{1}{k}$.

Then if $x \in A$ then η_m even and

$$f_k(x) = \sum_{\substack{n\leq k \\ \eta_n \text{ even}}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) =$$

$$= \left(\sum_{\substack{n\leq k \\ \eta_n \text{ even} \\ \eta_n<\eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) \right) + \left(\sum_{\substack{n\leq k \\ \eta_n \text{ even} \\ \eta_n>\eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) \right) +$$

$$+ f^k_{\eta_m}(x) - f^k_{\eta_m+1}(x).$$

The first sum is clearly 0 since $f^k_{\eta_n} \equiv 1$ on F_{η_m} if $\eta_m > \eta_n$. This is also true for the second one, since if $d(x, F_{\eta_n}) \geq \frac{1}{k}$ then $f^k_{\eta_n}(x) = 0$. Finally, $f_{\eta_m}(x) = 1$ and $f_{\eta_m+1}(x) = 0$, so $f_k(x) = 1$.

If $x \notin A$ then η_m is odd and

$$f_k(x) = \sum_{\substack{n \leq k \\ \eta_n \text{ even}}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) =$$

$$= \sum_{\substack{n \leq k \\ \eta_n \text{ even} \\ \eta_n < \eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x) + \sum_{\substack{n \leq k \\ \eta_n \text{ even} \\ \eta_n > \eta_m}} f^k_{\eta_n}(x) - f^k_{\eta_n+1}(x).$$

Now the previous argument gives $f_k(x) = 0$.

So $f_k \to \chi_A$ holds. Next we prove by induction on η that for every $\eta < \lambda$ and every $\varepsilon > 0$ we have

$$D^\eta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) \subset F_\eta.$$

This will clearly complete the proof.

For $\eta = 0$ we have

$$D^0_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) = X = F_0.$$

If η is a limit ordinal, the statement is clear, since the sequence of derivatives as well as $(F_\eta)_{\eta<\lambda}$ are continuous.

Now let $\eta = \theta + 1$ and $D^\theta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X) \subset F_\theta$. For some m we have $\theta = \eta_m$. Let $x \in F_{\eta_m} \setminus F_{\eta_m+1}$. Then it is enough to prove that $x \notin D^\eta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$. Let k be such that $d(x, F_{\eta_m+1}) \geq \frac{2}{k}$.

Whenever $d(x, y) < \frac{1}{k}$ and $y \in D^\theta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$ then $y \in F_{\eta_m} \setminus F_{\eta_m+1}$. From this, if $l_1, l_2 \geq k$ we have that $f^{l_1}_\eta(y) = f^{l_2}_\eta(y) = 1$ if $\eta \leq \eta_m$ and $f^{l_1}_\eta(y) = f^{l_2}_\eta(y) = 0$ if $\eta > \eta_m$. Hence $f_{l_1}(y) - f_{l_2}(y) = 0$.

So we have that the sequence f_k is eventually constant on a relative neighbourhood of x in F_{η_m}, therefore $x \notin D^\eta_{(f_k)_{k\in\mathbb{N}},\varepsilon}(X)$, which finishes the proof. □

Next we prove that $\gamma(f) \lesssim \alpha(f)$ for every step function f. We still need the following lemma.

Lemma 4.2.37. *If A and B are ambiguous sets then*

$$\alpha\left(\chi_{A\cap B}\right) \lesssim \max\left\{\alpha\left(\chi_A\right), \alpha\left(\chi_B\right)\right\}.$$

Proof. It is enough to prove this for β since the previous lemma and Theorem 4.2.24 yields that the ranks essentially agree on characteristic functions. Theorem 4.2.29 gives $\beta(\chi_A+\chi_B) \lesssim \max\{\beta(\chi_A), \beta(\chi_B)\}$, hence it suffices to prove that $\beta\left(\chi_{A\cap B}\right) \leq \beta(\chi_A+\chi_B)$. But this easily follows, since one can readily check that for every $\varepsilon < 1$ and F we have $D_{\chi_{A\cap B},\varepsilon}(F) \subseteq D_{\chi_A+\chi_B,\varepsilon}(F)$, finishing the proof. □

Now let f be a step function, so $f = \sum_{i=1}^{n} c_i \chi_{A_i}$, where the A_i's are disjoint ambiguous sets covering X, and we can also suppose that the c_i's form a strictly increasing sequence of real numbers.

Lemma 4.2.38. $\max_i\{\alpha(\chi_{A_i})\} \lesssim \alpha(f)$.

Proof. Let $H_i = \bigcup_{j=1}^{i} A_j$. By the definition of the rank α, for every i we have

$$\alpha(H_i, H_i^c) \leq \alpha(f). \tag{4.2.17}$$

This shows that $\alpha(\chi_{A_1}) \lesssim \alpha(f)$, and together with the previous lemma, for $i > 1$

$$\begin{aligned}\alpha(\chi_{A_i}) = \alpha(\chi_{H_i \setminus H_{i-1}}) = \alpha(\chi_{H_i \cap H_{i-1}^c}) &\lesssim \max\{\alpha(\chi_{H_i}), \alpha(\chi_{H_{i-1}^c})\} \\ &= \max\{\alpha(H_i, H_i^c), \alpha(H_{i-1}, H_{i-1}^c)\} \leq \alpha(f),\end{aligned}$$

where the last but one inequality follows from the above lemma and the last inequality from (4.2.17). □

Now we have

$$\gamma(f) \lesssim \max_i\{\gamma(\chi_{A_i})\} \approx \max_i\{\alpha(\chi_{A_i})\} \lesssim \alpha(f),$$

where we used Theorem 4.2.29, this theorem for characteristic functions and Lemma 4.2.38, proving the theorem for step functions.

In particular, $\alpha(f) \leq \beta(f) \leq \gamma(f)$ (Theorem 4.2.24) gives the following corollary.

Corollary 4.2.39. *If $f = \sum_{i=1}^{n} c_i \chi_{A_i}$, where the A_i's are disjoint ambiguous sets covering X and the c_i's are distinct then*

$$\alpha(f) \approx \max_i\{\alpha(\chi_{A_i})\}$$

and similarly for β and γ.

Now let f be an arbitrary bounded Baire class 1 function.

Lemma 4.2.40. *There is a sequence f_n of step functions converging uniformly to f, satisfying $\sup_n \alpha(f_n) \lesssim \alpha(f)$.*

Proof. Let $p_{n,k} = k/2^n$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The level sets $\{f \leq p_{n,k}\}$ and $\{f \geq p_{n,k+1}\}$ are disjoint $\mathbf{\Pi}_2^0$ sets, hence they can be separated by a $H_{n,k} \in \mathbf{\Delta}_2^0(X)$ (see e.g. [40, 22.16]). We can choose $H_{n,k}$ to satisfy $\alpha_1(H_{n,k}, H_{n,k}^c) \leq 2\alpha(f)$ using Proposition 4.2.12.

Since f is bounded, for fixed n there are only finitely many $k \in \mathbb{Z}$ for which $H_{n,k+1} \setminus H_{n,k} \neq \emptyset$. Set

$$f_n = \sum_{k \in \mathbb{Z}} p_{n,k} \cdot \chi_{H_{n,k+1} \setminus H_{n,k}}.$$

Now for each n, f_n is a step function with $|f - f_n| \leq 2^{n-1}$. Hence $f_n \to f$ uniformly. Since the level sets of a function f_n are of the form $H_{n,k}$ or $H_{n,k}^c$ for some $k \in \mathbb{Z}$, we have $\alpha(f_n) \leq 2\alpha(f)$, proving the lemma. □

Let f_n be a sequence of step functions given by this lemma. Using Proposition 4.2.34 and this theorem for step functions, we have $\gamma(f) \lesssim \sup_n \gamma(f_n) \lesssim \sup_n \alpha_n(f_n) \lesssim \alpha(f)$, completing the proof. $\square$

We have seen above that α is not essentially additive on the Baire class 1 functions but β and γ are, therefore α cannot essentially coincide with β or γ. However, in view of the above theorem the following question arises.

Question 4.2.41. *Does $\beta \approx \gamma$ hold for arbitrary Baire class 1 functions?*

Proposition 4.2.42. *If the sequence of Baire class 1 functions f_n converges uniformly to f then $\alpha(f) \lesssim \sup_n \alpha(f_n)$.*

Proof. If f is bounded (hence without loss of generality the f_n are also bounded) this is an easy consequence of Theorem 4.2.35 and Proposition 4.2.33.

For an arbitrary function g let $g' = \arctan \circ g$. It is easy to show that $\alpha(g') = \alpha(g)$ using Remark 4.2.9.

If the functions f and f_n are given such that $f_n \to f$ uniformly then $f'_n \to f'$ uniformly, and these are bounded functions, so we have $\alpha(f) = \alpha(f') \lesssim \sup_n \alpha(f'_n) = \sup_n \alpha(f_n)$. $\square$

4.3 Ranks on the Baire class ξ functions exhibiting strange phenomena

4.3.1 The separation rank and the linearised separation rank

The only rank out of the ones discussed above that has straightforward generalisation to the Baire class ξ case is the rank α_1. However, this generalisation does not answer Question 4.0.2, since, similarly to the original α_1, it is not linear. After discussing this, we will propose a very natural modification that transforms an arbitrary rank into a linear one, but we well see that this modified rank will be bounded in ω_1 for characteristic functions!

Definition 4.3.1. Let A and B be disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets. Then they can be separated by a $\mathbf{\Delta}^0_{\xi+1}$ set (see e.g. [40, 22.16]). Since every $\mathbf{\Delta}^0_{\xi+1}$ set is the transfinite difference of $\mathbf{\Pi}^0_{\xi}$ sets, A and B can be separated by the transfinite difference of such a sequence. Let $\alpha_\xi(A, B)$ denote the length of the shortest such sequence.

Definition 4.3.2. Let f be a Baire class ξ function, and $p < q \in \mathbb{Q}$. Then $\{f \leq p\}$ and $\{f \geq q\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets. Let the *separation rank* of f be

$$\alpha_\xi(f) = \sup_{\substack{p<q \\ p,q\in\mathbb{Q}}} \alpha_\xi(\{f \leq p\}, \{f \geq q\}).$$

Note that this really extends the definition of α_1.

Theorem 4.3.3. *For every $1 \leq \xi < \omega_1$ the rank α_ξ is unbounded in ω_1 on the characteristic Baire class ξ functions.*

Proof. Let $\mathcal{U} \in \mathbf{\Pi}^0_\xi(2^\omega \times X)$ be a universal set for $\mathbf{\Pi}^0_\xi(X)$ sets, that is, for every $F \subseteq X$, $F \in \mathbf{\Pi}^0_\xi(X)$ there exists a $y \in 2^\omega$ such that $\mathcal{U}^y = F$. For the existence of such a set see [40, 22.3]. Let us use the notation $\Gamma_\zeta(X)$ for the the family of sets $H \subseteq X$ satisfying $\alpha_\xi(H, H^c) < \zeta$. From [40, 22.27] we have $\Gamma_\zeta(X) \subseteq \mathbf{\Delta}^0_{\xi+1}(X)$. We will show that there exists a $\mathbf{\Delta}^0_{\xi+1}$ set for every $\zeta < \omega_1$ which is universal for the family of Γ_ζ sets. Since X is uncountable, there is a continuous embedding of 2^ω into X ([40, 6.5]), hence no universal set exists in $2^\omega \times X$ for the family of $\mathbf{\Delta}^0_{\xi+1}(X)$ sets (easy corollary of [40, 22.7]). This implies for every $\zeta < \omega_1$ that $\Gamma_\zeta \neq \mathbf{\Delta}^0_{\xi+1}$, hence the rank is really unbounded.

Let $p : \zeta \times \mathbb{N} \to \mathbb{N}$ be a bijection. For $\eta < \zeta$ and $y \in 2^\omega$ we define a $\phi(y)_\eta \in 2^\omega$ by $\phi(y)_\eta(n) = y(p(\eta, n))$. First we check that for a fixed $\eta < \zeta$ the map $y \mapsto \phi(y)_\eta$ is continuous. Let $U = \{x \in 2^\omega : x(0) = i_0, \ldots, x(n) = i_n\}$ be a set from the usual basis of 2^ω. The preimage of U is the set $\{y \in 2^\omega : \forall k \leq n \ \phi(y)_\eta(k) = i_k\} = \{y \in 2^\omega : \forall k \leq n \ y(p(\eta, k)) = i_k\}$, which is a basic open set, too. Now $\mathcal{U}_\eta = \{(y, x) : (\phi(y)_\eta, x) \in \mathcal{U}\}$ is a continuous preimage of a $\mathbf{\Pi}^0_\xi$ set, hence $\mathcal{U}_\eta \in \mathbf{\Pi}^0_\xi(2^\omega \times X)$ (see [40, 22.1]). Let

$$\begin{aligned}\mathcal{U}' = \{(y, x) \in 2^\omega \times X : {} & \text{the smallest ordinal } \eta \text{ such that } (y, x) \notin \mathcal{U}_\eta \text{ is odd,} \\ & \text{if such an } \eta \text{ exists, or no such } \eta \text{ exists and } \zeta \text{ is odd}\}.\end{aligned}$$

Now we check that $\mathcal{U}' \in \mathbf{\Delta}^0_{\xi+1}(2^\omega \times X)$. Let $\mathcal{V}_\eta = \bigcap_{\theta < \eta} \mathcal{U}_\theta$, then these sets form a continuous decreasing sequence of $\mathbf{\Pi}^0_\xi$ sets and it is easy to see that $\mathcal{U}'^c$ is the transfinite difference of the sequence $(\mathcal{V}_\eta)_{\eta < \zeta+1}$, hence $\mathcal{U}'^c \in \mathbf{\Delta}^0_{\xi+1}$, proving that $\mathcal{U}' \in \mathbf{\Delta}^0_{\xi+1}$, since the family of $\mathbf{\Delta}^0_{\xi+1}$ sets is closed under complements (see [40, 22.1]).

Now we show that $\mathcal{U}'$ is universal. For a set $H \in \Gamma_\zeta(X)$ there is a sequence $(z_\eta)_{\eta<\zeta}$ in 2^ω, such that H is the transfinite difference of the sets $\mathcal{U}^{z_\eta}$. For every sequence $(z_\eta)_{\eta<\zeta}$ we can find a $y \in 2^\omega$ such that $\phi(y)_\eta = z_\eta$, namely $y : p(\eta, n) \mapsto z_\eta(n)$ makes sense (since p is a bijection), and works. Consequently, for H there is a $y \in 2^\omega$, such that H is the transfinite difference of the sets $\mathcal{U}^{z_\eta} = \mathcal{U}^{\phi(y)_\eta} = (\mathcal{U}_\eta)^y$. It is easy to see that if H is the transfinite difference of the sequence $((\mathcal{U}_\eta)^y)_{\eta<\zeta}$ then

$$\begin{aligned}H = \{x \in X : {} & \text{the smallest ordinal } \eta \text{ such that } x \notin (\mathcal{U}_\eta)^y \text{ is odd,} \\ & \text{if such an } \eta \text{ exists, or no such } \eta \text{ exists and } \zeta \text{ is odd}\},\end{aligned}$$

hence $H = \mathcal{U}'^y$. □

Corollary 4.3.4. *For every $1 \leq \xi < \omega_1$, every non-empty perfect set $P \subseteq X$ and every ordinal $\zeta < \omega_1$ there is a characteristic function $\chi_A \in \mathcal{B}_\xi(X)$ with $A \subseteq P$ and $\alpha_\xi(\chi_A) \geq \zeta$.*

Proof. Since P is perfect, it is an uncountable Polish space with the subspace topology, hence the rank α_ξ is unbounded on the characteristic Baire class ξ functions defined on

P by the previous theorem. Hence we can take a characteristic function $f' \in \mathcal{B}_\xi(P)$ with $\alpha_\xi(f') \geq \zeta$, and set

$$f(x) = \begin{cases} f'(x) & \text{if } x \in P \\ 0 & \text{if } x \in X \setminus P. \end{cases}$$

It is easy to see that $f \in \mathcal{B}_\xi(X)$, hence it is enough to prove that $\alpha_\xi(f) \geq \zeta$.

For this, it is enough to prove that $\alpha_\xi(\{f' \leq p\}, \{f' \geq q\}) \leq \alpha_\xi(\{f \leq p\}, \{f \geq q\})$ for every pair of rational numbers $p < q$. For this, let $H \in \mathbf{\Delta}^0_{\xi+1}(X)$ where $\{f \leq p\} \subseteq H \subseteq \{f \geq q\}^c$ and H is the transfinite difference of the sets $(F_\eta)_{\eta<\lambda}$ with $\lambda = \alpha_\xi(\{f \leq p\}, \{f \geq q\})$ and $F_\eta \in \mathbf{\Pi}^0_\xi(X)$ for every $\eta < \lambda$.

Let $H' = P \cap H$ and for every $\eta < \lambda$ let $F'_\eta = P \cap F_\eta$. It is easy to see that H' separates the level sets $\{f' \leq p\}$ and $\{f' \geq q\}$ and H' is the transfinite difference of the sets $(F'_\eta)_{\eta<\lambda}$. And since $H' \in \mathbf{\Delta}^0_{\xi+1}(P)$ and $F'_\eta \in \mathbf{\Pi}^0_\xi(P)$ for every $\eta < \lambda$ ([40, 22.A]), we have the desired inequality $\alpha_\xi(\{f' \leq p\}, \{f' \geq q\}) \leq \alpha_\xi(\{f \leq p\}, \{f \geq q\})$. Thus the proof is complete. □

The main disadvantage of this rank is that the construction of Remark 4.2.30 easily yields that the rank does not behave nicely under linear operations. We leave the easy proof of the next statement to the reader.

Proposition 4.3.5. *Let $1 \leq \xi < \omega_1$. Then α_ξ is not essentially linear, actually not even essentially additive.*

However, there is a natural way to make a rank linear.

Definition 4.3.6. For an $f \in \mathcal{B}_\xi$, let

$$\alpha'_\xi(f) = \min\{\max\{\alpha_\xi(f_1), \dots, \alpha_\xi(f_n)\} : n \in \omega, f_1, \dots, f_n \in \mathcal{B}_\xi,\ f = f_1 + \dots + f_n\}.$$

It can be easily seen that α'_ξ is now linear, but we do not know whether it is still unbounded in ω_1.

Question 4.3.7. *Let $1 \leq \xi < \omega_1$. Is α'_ξ unbounded in ω_1?*

We have the following partial result, which is a very strong indication that the answer to this question is in the negative, since in every single case when we can show that a rank is unbounded it is actually unbounded on the characteristic functions.

Theorem 4.3.8. *If $1 \leq \xi < \omega_1$ and f is a* characteristic *Baire class ξ function then $\alpha'_\xi(f) \leq 2$.*

Proof. Let us call a function f a semi-Borel class ξ function if the level sets $\{f < c\}$ are in $\mathbf{\Sigma}^0_\xi$ for every $c \in \mathbb{R}$. Note that then the level sets $\{f > c\}$ are in $\mathbf{\Sigma}^0_{\xi+1}$, hence $f \in \mathcal{B}_\xi$.

We first show that a semi-Borel class ξ function has α_ξ rank at most 2. Let $p < q$ be a pair of rational numbers. The level set $\{f \geq q\} \in \mathbf{\Pi}^0_\xi(X)$, hence the transfinite difference of the sequence $F_0 = X, F_1 = \{f \geq q\}$ separates the level sets $\{f \leq p\}$ and $\{f \geq q\}$.

Now using the same idea as in Remark 4.2.30, it is clear that every characteristic Baire class ξ function can be written as the difference of two semi-Borel class ξ functions, completing the proof of this theorem. □

The following question is very closely related to Question 4.3.7.

Question 4.3.9. *Let $1 \leq \xi < \omega_1$ and let f_n and f be Baire class ξ functions such that $f_n \to f$ uniformly. Does this imply that $\alpha'_\xi(f) \lesssim \sup_n \alpha'_\xi(f_n)$?*

Remark 4.3.10. An affirmative answer to this question would provide a negative answer to Question 4.3.7. Indeed, it is not hard to show that α'_ξ is bounded for step functions, and hence, by taking uniform limit, for every bounded function. Then one can check that the rank of an arbitrary function f equals to the rank of the bounded function $\arctan \circ f$, hence α'_ξ is bounded.

4.3.2 Limit ranks

In this section we apply an even more natural approach to define ranks on the Baire class ξ functions starting from an arbitrary rank on the Baire class 1 functions. Surprisingly, they will all turn out to be bounded in ω_1.

Definition 4.3.11. Let ρ be a rank on the Baire class 1 functions. We inductively define a rank $\overline{\rho}_\xi$ on the Baire class ξ functions. First, let $\overline{\rho}_1 = \rho$. For a successor ordinal $\xi + 1$ and a Baire class $\xi + 1$ function f let

$$\overline{\rho}_{\xi+1}(f) = \min\left\{\sup_n \overline{\rho}_\xi(f_n) : f_n \to f,\ f_n \text{ is of Baire class } \xi\right\}.$$

Finally, for a limit ordinal ξ and a Baire class ξ function f let

$$\overline{\rho}_\xi(f) = \min\Bigg\{\sup_n \overline{\rho}_{\xi_n}(f_n) : f_n \to f,\ f_n \text{ is of Baire class } \xi_n,\ \xi_n < \xi, \\ f_n \text{ is not of Baire class } \zeta \text{ if } \zeta < \xi_n\Bigg\}.$$

Surprisingly, the ranks $\overline{\alpha}_\xi$, $\overline{\beta}_\xi$ and $\overline{\gamma}_\xi$ will all be bounded for $\xi \geq 2$.

Theorem 4.3.12. *If $2 \leq \xi < \omega_1$ then $\overline{\alpha}_\xi \leq \overline{\beta}_\xi \leq \overline{\gamma}_\xi \leq \omega$.*

Proof. It is enough to prove the theorem for $\xi = 2$. Let Φ be a class of real valued functions on X. As in [34], we say that Φ is *ordinary* if it contains the constant functions and if $f, g \in \Phi$ then $\max(f,g)$, $\min(f,g)$, $f+g$, $f-g$, fg and f/g (if g is nowhere zero) are all in Φ. An ordinary class of functions is called *complete* if it is closed under uniform limits.

For a class of functions Φ, we denote by Φ^p the set of functions that are pointwise limits of functions from Φ. We denote the pair of families of level sets of functions in Φ by $\mathbf{P}(\Phi)$, that is,

$$\mathbf{P}(\Phi) = \left(\left\{\{f > c\} : f \in \Phi, c \in \mathbb{R}\right\}, \left\{\{f \geq c\} : f \in \Phi, c \in \mathbb{R}\right\}\right).$$

If $\mathbf{P} = (\mathcal{M}, \mathcal{N})$ is a pair of systems of sets then we denote the class of functions whose levels sets are in $\mathbf{P}$ by $\Phi(\mathbf{P})$, that is,

$$\Phi(\mathbf{P}) = \left\{f : X \to \mathbb{R} \mid \forall c \in \mathbb{R}\ \{f > c\} \in \mathcal{M}, \{f \geq c\} \in \mathcal{N}\right\}.$$

Now we state three theorems based on results in [34].

Theorem 4.3.13. *If a class of functions Φ is ordinary then Φ^p is ordinary and complete.*

Theorem 4.3.14. *If a class of functions Φ is ordinary and $\mathbf{P}(\Phi) = (\mathcal{M}, \mathcal{N})$ then $\mathbf{P}(\Phi^p) = (\mathcal{N}_{\delta\sigma}, \mathcal{M}_{\sigma\delta})$.*

Theorem 4.3.15. *If a class of functions Φ is complete and ordinary then $\Phi = \Phi(\mathbf{P}(\Phi))$.*

Theorem 4.3.13 is shown in [34, §41. IV.], Theorem 4.3.14 is an easy corollary of [34, §41. V., VI.] and Theorem 4.3.15 is shown in [34, §41. VIII.].

Now let Φ consist of the Baire class 1 functions of the form

$$\sum_{i=1}^{n} c_i \chi_{H_i},$$

where H_i is in the algebra $\mathcal{A}$ generated by the open sets (an algebra is a family closed under finite unions and complements). It is easy to check that $\mathcal{A}$ contains exactly the sets that can be written as the finite disjoint union of sets of the form $F \cap G$, where F is closed and G is open. Indeed, the intersection of two such set is of the same form, and the complement of such a set is

$$\left(\bigcup_{i=0}^{n-1} (F_i \cap G_i)\right)^c = \bigcap_{i=0}^{n-1} (F_i \cap G_i)^c = \bigcap_{i=0}^{n-1} (F_i^c \cup G_i^c) =$$

$$\bigcup \left\{ \bigcap_{i=0}^{n-1} F_i^{a(i)} \cap \bigcap_{i=0}^{n-1} G_i^{b(i)} : a, b \in 2^n, \forall i < n \text{ at least one of } a(i) \text{ and } b(i) \text{ is } 1 \right\},$$

where for a set H, $H^0 = H$ and $H^1 = H^c$, and the last equality holds, since a point x is contained in either of the two sets in question iff for every $i < n$ it is contained in at least one of F_i^c and G_i^c. Now we check that the sets in the union are disjoint. Without loss of generality we have two terms with distinct a's, so $a(i) = 0$ and $a'(i) = 1$ for a suitable i. But then the term belonging to a is a subset of F_i and the other one is a subset of F_i^c, proving disjointness.

An easy consequence of these observations is that Φ is ordinary.

Lemma 4.3.16. *$\gamma(f) \leq \omega$ for every $f \in \Phi$.*

Proof. First we prove that $\gamma(\chi_F) \leq 2$ for every closed set F. Let F be a closed set, and define $f_n(x) = 1 - \min\{1, n \cdot d(x, F)\}$. It is easy to check that $f_n \to \chi_F$ pointwise. We now show that $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq 2$ for every $\varepsilon > 0$, which will imply $\gamma(\chi_F) \leq 2$. Fix $\varepsilon > 0$. If $x \notin F$ then x has a neighbourhood U such that $d(U, F) > 0$ and then if we fix an $N > \frac{1}{d(U,F)}$ then $f_n(y) = 0$ for every $y \in U$ and $n \geq N$, therefore $\omega((f_n)_{n\in\mathbb{N}}, x, X) = 0$. This implies $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq F$. But $f_n|_F \equiv 1$ for every n, hence if $x \in F$ then $\omega((f_n)_{n\in\mathbb{N}}, x, F) = 0$, therefore $D^2_{(f_n)_{n\in\mathbb{N}},\varepsilon}(X) \subseteq D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \emptyset$, proving $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq 2$.

It is easy to check that $\gamma(f) = \gamma(1 - f)$ for every $f \in \mathcal{B}_1$. This implies that $\gamma(\chi_G) \leq 2$ for every open set G, since $\chi_G = 1 - \chi_{X\setminus G}$.

Now, let $H = F \cap G$, where F is closed and G is open. We show that $\gamma(\chi_H) \leq \omega$. By Theorem 4.2.29 there exists a sequence f_n of continuous functions with $f_n \to \chi_F + \chi_G$ and $\gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq \omega$ for every $\varepsilon > 0$. Define $f'_n = \max\{0, f_n - 1\}$. Then it is easy to check that $f'_n \to \chi_H$ and $\gamma((f'_n)_{n\in\mathbb{N}}, \varepsilon) \leq \gamma((f_n)_{n\in\mathbb{N}}, \varepsilon) \leq \omega$ for every $\varepsilon > 0$.

Since any $H \in \mathcal{A}$ is a finite disjoint union of sets of the form $F \cap G$, the above paragraph shows that $\chi_H = \chi_{H_0} + \cdots + \chi_{H_n}$, where $\gamma(\chi_{H_i}) \leq \omega$. But then Theorem 4.2.29 yields that $\gamma(\chi_H) \leq \omega$. Then applying Theorem 4.2.29 once again we obtain that $\gamma(f) \leq \omega$ for every $f \in \Phi$. $\square$

Now we turn to the proof of the theorem. By Theorem 4.2.24 and the previous lemma, it is enough to show that Φ^p equals the family of Baire class 2 functions. Since every $f \in \Phi$ is of Baire class 1, we have that Φ^p is a subclass of the Baire class 2 functions.

For the converse, let us define $\mathcal{M}$ and $\mathcal{N}$ by $\mathbf{P}(\Phi) = (\mathcal{M}, \mathcal{N})$. By the definition of Φ, $\mathcal{M}$ and $\mathcal{N}$ both contain the open and closed sets. By Theorem 4.3.14 $\mathbf{P}(\Phi^p) = (\mathcal{N}_{\delta\sigma}, \mathcal{M}_{\sigma\delta})$, hence $\mathbf{\Sigma}^0_3 \subseteq \mathcal{N}_{\delta\sigma}$ and $\mathbf{\Pi}^0_3 \subseteq \mathcal{M}_{\sigma\delta}$. And by Theorem 4.3.13 and Theorem 4.3.15 $\Phi^p = \Phi(\mathbf{P}(\Phi^p)) = \Phi(\mathcal{N}_{\delta\sigma}, \mathcal{M}_{\sigma\delta}) \supseteq \Phi(\mathbf{\Sigma}^0_3, \mathbf{\Pi}^0_3) = \mathcal{B}_2$, finishing the proof. $\square$

4.3.3 Partition ranks

The following well known fact also gives rise to a very natural rank on the Baire class ξ functions. However, this also turns out to be bounded.

Proposition 4.3.17. *A function f is of Baire class ξ if and only if for every $\varepsilon > 0$ there exists a function g of the form $g = \sum_{n\in\omega} c_n \cdot \chi_{H_n}$, where $H_n \in \mathbf{\Delta}^0_{\xi+1}(X)$, the H_n's form a partition of X and $|f(x) - g(x)| \leq \varepsilon$ for every $x \in X$. Moreover, if f is bounded then each set H_n can be chosen to be empty for all but finitely many $n \in \omega$.*

Proof. If f is of Baire class ξ then for a fixed $\varepsilon > 0$ let the numbers p_n be defined by $p_n = n \cdot \frac{\varepsilon}{2}$ for every $n \in \mathbb{Z}$. The sets $\{f \leq p_n\}$ and $\{f \geq p_{n+1}\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they can be separated by a set $A_n \in \mathbf{\Delta}^0_{\xi+1}$. Now let $H_n = A_n \setminus A_{n-1}$. Note that if f is bounded then $H_n = \emptyset$ for all but finitely many $n \in \omega$. These sets form a partition, and with $g = \sum_{n\in\mathbb{Z}} p_n \cdot \chi_{H_n}$ the proof of the first direction is complete.

For the other one, note that the function g is of Baire class ξ, hence f is the uniform limit of Baire class ξ functions, implying that f is of Baire class ξ (see e.g. [40, 24.4]). □

Definition 4.3.18. Let f be a Baire class ξ function and let the *partition rank* of f be

$$\delta(f) = \sup_{\varepsilon>0} \min \Bigg\{ \sup_{n\in\omega} \alpha_\xi(H_n, H_n^c) : \; H_n \in \mathbf{\Delta}^0_{\xi+1}, \; \bigcup_{n\in\omega} H_n = X,$$
$$H_n \cap H_m = \emptyset \; (n \neq m), \; \exists (c_n)_{n\in\omega} \left| f - \sum_{n\in\omega} c_n \cdot \chi_{H_n} \right| \leq \varepsilon \Bigg\}.$$

Proposition 4.3.19. *$\delta(f) \leq 4$ for every Baire class ξ function f.*

Proof. Fix $\varepsilon > 0$. Obtain a function of the form $\sum_{n\in\omega} c_n \cdot \chi_{H_n}$ as in the above proposition. It is enough to prove that every H_n has a further partition into a sequence of sets $H_{n,k} \in \mathbf{\Delta}^0_{\xi+1}$ with $\alpha_\xi(H_{n,k}, H^c_{n,k}) \leq 4$.

But this is easy, since H_n can be written as the transfinite difference of $\mathbf{\Pi}^0_\xi$ sets, so H_n is obtained as the countable disjoint union of sets of the form $F_\eta \setminus F_{\eta+1}$ with $F_\eta, F_{\eta+1} \in \mathbf{\Pi}^0_\xi$, and the α_ξ rank of $F_\eta \setminus F_{\eta+1}$ at most 4, as the sequence $(X, X, F_\eta, F_{\eta+1})$ shows. □

Now we focus our attention on finite partitions and investigate the resulting rank, which we can only define for bounded functions.

Definition 4.3.20. Let f be a bounded Baire class ξ function and let the *finite partition rank* of f be

$$\delta_{fin}(f) = \sup_{\varepsilon>0} \min \Bigg\{ \sup_{n\leq N} \alpha_\xi(H_n, H_n^c) : N \in \omega, H_n \in \mathbf{\Delta}^0_{\xi+1} (n \leq N), \bigcup_{n\leq N} H_n = X,$$
$$H_n \cap H_m = \emptyset \; (n, m \leq N, \; n \neq m), \; \exists (c_n)_{n\leq N} \left| f - \sum_{n\leq N} c_n \cdot \chi_{H_n} \right| \leq \varepsilon \Bigg\}.$$

Theorem 4.3.21. *$\delta_{fin}(f) \approx \alpha_\xi(f)$ for every bounded Baire class ξ function f.*

Proof. Let f be an arbitrary bounded Baire class ξ function. First we prove that $\delta_{fin} \lesssim \alpha_\xi(f)$. For a fixed $\varepsilon > 0$ let the numbers p_n be defined by $p_n = n \cdot \frac{\varepsilon}{2}$ for every $n \in \mathbb{Z}$. The sets $\{f \leq p_n\}$ and $\{f \geq p_{n+1}\}$ are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they can be separated by a set $A_n \in \mathbf{\Delta}^0_{\xi+1}$ with $\alpha_\xi(A_n, A_n^c) \leq \alpha_\xi(f)$. Now let $H_n = A_n \setminus A_{n-1}$. Since f is bounded, $H_n = \emptyset$ for all but finitely many $n \in \omega$. Clearly, these sets form a partition, and $g = \sum_{n\in\mathbb{Z}} p_n \cdot \chi_{H_n}$ is ε-close to f.

We will prove in Corollary 4.4.18 below that α_ξ is essentially linear for bounded functions. Therefore we obtain $\alpha_\xi(H_n, H_n^c) = \alpha_\xi(\chi_{H_n}) = \alpha_\xi(\chi_{A_n} - \chi_{A_{n-1}}) \lesssim \max\{\alpha_\xi(\chi_{A_n}), \alpha_\xi(\chi_{A_{n-1}})\} = \max\{\alpha_\xi(A_n, A_n^c), \alpha_\xi(A_{n-1}, A_{n-1}^c)\} \leq \alpha_\xi(f)$, proving $\delta_{fin} \lesssim \alpha_\xi(f)$.

Now we prove the other direction. Let $p < q$ be arbitrary rational numbers, it is enough to prove that there is a set $H \in \mathbf{\Delta}^0_{\xi+1}$ separating the level sets $\{f \leq p\}$ and $\{f \geq q\}$ with $\alpha_\xi(H, H^c) \leq \delta_{fin}(f)$. Now set $\varepsilon = \frac{q-p}{2}$. From the definition of δ_{fin}, we can find a finite partition $X = H_0 \cup \dots \cup H_N$ into disjoint $\mathbf{\Delta}^0_{\xi+1}$ sets and $c_n \in \mathbb{R}$ with $g = \sum_{n=0}^N c_n \cdot \chi_{H_n}$ satisfying $|f - g| < \varepsilon$ and $\alpha_\xi(H_n, H_n^c) \leq \delta_{fin}(f)$ for $n \leq N$.

Let $A = \{n \leq N : H_n \cap \{f \leq p\} \neq \emptyset\}$ and $H = \bigcup_{n \in A} H_n$. Clearly, $\{f \leq p\} \subseteq H$. Moreover, no H_n can intersect both $\{f \leq p\}$ and $\{f \geq q\}$, since g is constant on H_n and $|f - g| < \varepsilon = \frac{q-p}{2}$. Therefore $H \cap \{f \geq q\} = \emptyset$. Using the essential linearity of α_ξ for bounded functions again we obtain $\alpha_\xi(H, H^c) = \alpha_\xi(\chi_H) \lesssim \max\{\alpha_\xi(\chi_{H_n}) : n \in A\} = \max\{\alpha_\xi(H_n, H_n^c) : n \in A\} \leq \delta_{fin}(f)$, completing the proof. □

4.4 Ranks answering Questions 4.0.1 and 4.0.2

In this section we finally show that there actually exist ranks with very nice properties. Throughout the section, let $1 \leq \xi < \omega_1$ be fixed.

Let f be of Baire class ξ. Let

$$T_{f,\xi} = \{\tau' : \tau' \supseteq \tau \text{ Polish}, \tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau), f \in \mathcal{B}_1(\tau')\}.$$

So $T_{f,\xi}$ is the set of those Polish refinements of the original topology that are subsets of the $\mathbf{\Sigma}^0_\xi$ sets turning f to a Baire class 1 function.

Remark 4.4.1. Clearly, $T_{f,1} = \{\tau\}$ for every Baire class 1 function f.

In order to show that the ranks we are about to construct are well-defined, we need the following proposition.

Proposition 4.4.2. *$T_{f,\xi} \neq \emptyset$ for every Baire class ξ function f.*

Proof. By the previous remark we may assume $\xi \geq 2$. For every rational p the level sets $\{f \leq p\}$ and $\{f \geq p\}$ are $\mathbf{\Pi}^0_{\xi+1}$ sets, hence they are countable intersections of $\mathbf{\Sigma}^0_\xi$ sets. In turn, these $\mathbf{\Sigma}^0_\xi$ sets are countable unions of sets from $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau)$. Clearly, $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(\tau) \subseteq \mathbf{\Delta}^0_\xi$ for $\xi \geq 2$. By Kuratowski's theorem [40, 22.18], there exists a Polish refinement $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$ of τ for which all these countable many $\mathbf{\Delta}^0_\xi$ sets are in $\mathbf{\Delta}^0_1(\tau')$. Then for every rational p the level sets are now $\mathbf{\Pi}^0_2(\tau')$ sets, and the same holds for irrational numbers too, since these level sets can be written as countable intersection of rational level sets, proving $T_{f,\xi} \neq \emptyset$. □

Now similarly to the limit ranks, we define a rank on the Baire class ξ functions starting from an arbitrary rank on the Baire class 1 functions.

Definition 4.4.3. Let ρ be a rank on the Baire class 1 functions. Then for a Baire class ξ function f let

$$\rho^*_\xi(f) = \min_{\tau' \in T_{f,\xi}} \rho_{\tau'}(f), \tag{4.4.1}$$

where $\rho_{\tau'}(f)$ is just the ρ rank of f in the τ' topology.

Remark 4.4.4. From Remark 4.4.1 it is clear that $\rho^*_1 = \rho$ for every ρ.

Proposition 4.4.5. *Let ρ and η be ranks on the Baire class 1 functions. If $\rho = \eta$, or $\rho \le \eta$, or $\rho \approx \eta$, or $\rho \lesssim \eta$ then $\rho^*_\xi = \eta^*_\xi$, or $\rho^*_\xi \le \eta^*_\xi$, or $\rho^*_\xi \approx \eta^*_\xi$, or $\rho^*_\xi \lesssim \eta^*_\xi$, respectively. Moreover, the same implications hold relative to the class of bounded functions.*

Proof. The statement for $=$ and $\le$ is immediate from the definitions, and the case of $\approx$ obviously follows from the case $\lesssim$, so it suffices to prove this latter case only. So assume $\rho \lesssim \eta$ (or $\rho \lesssim \eta$ on the bounded Baire class 1 functions). Choose an optimal $\tau' \in T_{f,\xi}$ for η, that is, $\eta^*_\xi(f) = \eta_{\tau'}(f)$. Then $\rho^*_\xi(f) \le \rho_{\tau'}(f) \lesssim \eta_{\tau'}(f) = \eta^*_\xi(f)$, completing the proof. □

Then the following two corollaries are immediate from Theorem 4.2.24, and Theorem 4.2.35.

Corollary 4.4.6. $\alpha^*_\xi \le \beta^*_\xi \le \gamma^*_\xi$.

Corollary 4.4.7. *$\alpha^*_\xi(f) \approx \beta^*_\xi(f) \approx \gamma^*_\xi(f)$ for every bounded Baire class ξ function f.*

Similarly to Question 4.2.41 (the case of Baire class 1 functions), we do not know whether $\beta^*_\xi(f) \approx \gamma^*_\xi(f)$ holds for arbitrary Baire class ξ functions.

Question 4.4.8. *Does $\beta^*_\xi(f) \approx \gamma^*_\xi(f)$ hold for every Baire class ξ function?*

Note that by repeating the argument of Remark 4.2.30 one can show that α^*_ξ differs from β^*_ξ and γ^*_ξ. It is easy to see that an affirmative answer to Question 4.2.41 would imply an affirmative answer to the last question, however, the other direction is not clear.

Theorem 4.4.9. *If X is a Polish group then the ranks α^*_ξ, β^*_ξ and γ^*_ξ are translation invariant.*

Proof. Note first that for a Baire class ξ function f and $x_0 \in X$ the functions $f \circ L_{x_0}$ and $f \circ R_{x_0}$ are also of Baire class ξ. We prove the statement only for the rank α^*_ξ, because an analogous argument works for the ranks β^*_ξ and γ^*_ξ.

Let f be a Baire class ξ function and $x_0 \in X$, first we prove that $\alpha^*_\xi(f) \ge \alpha^*_\xi(f \circ R_{x_0})$. Let $\tau' \in T_{f,\xi}$ be arbitrary and consider the topology $\tau'' = \{U \cdot x_0^{-1} : U \in \tau'\}$. The map $\phi : x \mapsto x \cdot x_0^{-1}$ is a homeomorphism between the spaces (X, τ') and (X, τ''), satisfying $f(x) = (f \circ R_{x_0})(\phi(x))$. From this it is clear that $\tau'' \in T_{f \circ R_{x_0},\xi}$ and since the definition of the rank α depends only on the topology of the space, we have $\alpha_{\tau'}(f) = \alpha_{\tau''}(f \circ R_{x_0})$. Since $\tau' \in T_{f,\xi}$ was arbitrary, the fact that $\alpha^*_\xi(f) \ge \alpha^*_\xi(f \circ R_{x_0})$ easily follows.

Repeating the argument with the function $f \circ R_{x_0}$ and element x_0^{-1}, we have $\alpha_\xi^*(f \circ R_{x_0}) \geq \alpha_\xi^*(f \circ R_{x_0} \circ R_{x_0^{-1}}) = \alpha_\xi^*(f)$, hence $\alpha_\xi^*(f) = \alpha_\xi^*(f \circ R_{x_0})$. For the function $f \circ L_{x_0}$ we can do same using the topology $\tau'' = \{x_0^{-1} \cdot U : U \in \tau'\}$ and homeomorphism $\phi : x \mapsto x_0^{-1} \cdot x$ yielding $\alpha_\xi^*(f) = \alpha_\xi^*(f \circ L_{x_0})$, finishing the proof. □

Theorem 4.4.10. *If f is a Baire class ξ function and $F \subseteq X$ is a closed set then $f \cdot \chi_F$ is of Baire class ξ and $\alpha_\xi^*(f \cdot \chi_F) \leq 1 + \alpha_\xi^*(f)$, $\beta_\xi^*(f \cdot \chi_F) \leq 1 + \beta_\xi^*(f)$ and $\gamma_\xi^*(f \cdot \chi_F) \leq 1 + \gamma_\xi^*(f)$.*

Proof. Examining the level sets of the function $f \cdot \chi_F$, it is easy to check that it is of Baire class ξ.

Now let $\tau' \in T_{f,\xi}$ be arbitrary. Clearly, $f \cdot \chi_F$ is of Baire class 1 with respect to τ', and by Proposition 4.2.28 we have $\alpha_{\tau'}(f \cdot \chi_F) \leq 1 + \alpha_{\tau'}(f)$ for every $\tau' \in T_{f,\xi}$, hence $\alpha_\xi^*(f \cdot \chi_F) \leq 1 + \alpha_\xi^*(f)$. The other two inequalities follow similarly. □

Proposition 4.4.11. *If f is a Baire class ζ function with $\zeta < \xi$ then $\alpha_\xi^*(f) = \beta_\xi^*(f) = \gamma_\xi^*(f) = 1$.*

Proof. Using Proposition 4.2.27, it is enough to show that there exists a topology $\tau' \in T_{f,\xi}$ such that $f : (X, \tau') \to \mathbb{R}$ is continuous, and this is clear from [40, 24.5]. □

Next we prove a useful lemma, and then investigate further properties of the ranks α_ξ^*, β_ξ^* and γ_ξ^*.

Lemma 4.4.12. *For every n let τ_n be a Polish refinement of τ with $\tau_n \subseteq \mathbf{\Sigma}_\xi^0(\tau)$. Then there exists a common Polish refinement τ' of the τ_n's also satisfying $\tau' \subseteq \mathbf{\Sigma}_\xi^0(\tau)$.*

Proof. The case $\xi = 1$ is again trivial, so we may assume $\xi \geq 2$. Take a base $\{G_n^k : k \in \mathbb{N}\}$ for τ_n. Since these sets are in $\mathbf{\Sigma}_\xi^0(\tau)$, they can be written as the countable unions of sets from $\bigcup_{\eta<\xi} \mathbf{\Pi}_\eta^0(\tau)$. Clearly, $\bigcup_{\eta<\xi} \mathbf{\Pi}_\eta^0(\tau) \subseteq \mathbf{\Delta}_\xi^0$ for $\xi \geq 2$. As above, by Kuratowski's theorem [40, 22.18], we have a Polish topology τ', for which these countably many $\mathbf{\Delta}_\xi^0(\tau)$ sets are in $\mathbf{\Delta}_1^0(\tau')$ satisfying $\tau' \subseteq \mathbf{\Sigma}_\xi^0(\tau)$. This τ' works. □

Lemma 4.4.13. *If $\tau' \subseteq \tau''$ are two Polish topologies with $f \in \mathcal{B}_1(\tau')$ then $f \in \mathcal{B}_1(\tau'')$, moreover, $\beta_{\tau'}(f) \geq \beta_{\tau''}(f)$ and $\gamma_{\tau'}(f) \geq \gamma_{\tau''}(f)$.*

Proof. To prove that $f \in \mathcal{B}_1(\tau'')$ note that the level sets $\{f < c\}, \{f > c\} \in \mathbf{\Sigma}_2^0(\tau')$, hence $\{f < c\}, \{f > c\} \in \mathbf{\Sigma}_2^0(\tau'')$, so $f \in \mathcal{B}_1(\tau'')$.

Now recall the definition of the derivative defining β:

$$\omega(f, x, F) = \inf \left\{ \sup_{x_1, x_2 \in U \cap F} |f(x_1) - f(x_2)| : U \text{ open}, x \in U \right\},$$

$$D_{f,\varepsilon}(F) = \{x \in F : \omega(f, x, F) \geq \varepsilon\}.$$

Let us now fix f and $\varepsilon > 0$ and let us denote the derivative $D_{f,\varepsilon}$ with respect to the topology τ' by $D_{\tau'}$, and with respect to the topology τ'' by $D_{\tau''}$. By Proposition 4.2.5 it is enough to prove that $D_{\tau''}(F) \subseteq D_{\tau'}(F)$ for every closed set $F \subseteq X$.

For this it is enough to show that $\omega_{\tau''}(f,x,F) \le \omega_{\tau'}(f,x,F)$ for every $x \in F$ where $\omega_{\tau'}(f,x,F)$ is the oscillation with respect to the topology τ'. And this is clear, since in the case of τ'', the infimum in the definition goes through more open set containing x, hence the resulting oscillation will be less.

For the rank γ, we proceed similarly. First we recall the definition of γ:

$$\omega((f_n)_{n\in\mathbb{N}},x,F) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup \left\{|f_m(y) - f_n(y)| : n,m \ge N,\ y \in U \cap F\right\},$$

$$D_{(f_n)_{n\in\mathbb{N}},\varepsilon}(F) = \{x \in F : \omega((f_n)_{n\in\mathbb{N}},x,F) \ge \varepsilon\},$$

$$\gamma(f) = \min\left\{\sup_{\varepsilon>0} \gamma((f_n)_{n\in\mathbb{N}},\varepsilon) : \forall n\ f_n \text{ is continuous and } f_n \to f \text{ pointwise}\right\}.$$

Let us fix a sequence $(f_n)_{n\in\mathbb{N}}$ of τ'-continuous (hence also τ''-continuous) functions converging pointwise to f, and also fix $\varepsilon > 0$. Let us denote the derivative $D_{(f_n)_{n\in\mathbb{N}},\varepsilon}$ with respect to τ' by $D_{\tau'}$ and with respect to τ'' by $D_{\tau''}$. Again, by Proposition 4.2.5 it is enough to prove that $D_{\tau''}(F) \subseteq D_{\tau'}(F)$ for every closed set $F \subseteq X$. And similarly to the previous case it is enough to prove that the oscillation $\omega((f_n)_{n\in\mathbb{N}},x,F)$ with respect to the topology τ'' is at most the oscillation with respect to τ', but this is clear, since, as before, the infimum goes through more open set in the case of τ''. $\square$

Theorem 4.4.14. *The ranks β^*_ξ and γ^*_ξ are essentially linear.*

Proof. We only consider β^*_ξ, since the proof for the rank γ^*_ξ is completely analogous.

It is easy to see that $\beta^*_\xi(cf) = \beta^*_\xi(f)$ for every $c \in \mathbb{R}\setminus\{0\}$, hence it suffices to show that β^*_ξ is essentially additive.

For f and g let τ_f and τ_g be such that $\beta_{\tau_f}(f) = \beta^*_\xi(f)$ and $\beta_{\tau_g}(g) = \beta^*_\xi(g)$. Using Lemma 4.4.12 we have a common refinement τ' of τ_f and τ_g with $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$. Now $f,g \in \mathcal{B}_1(\tau')$, so $f+g \in \mathcal{B}_1(\tau')$, hence $\tau' \in T_{f+g,\xi}$. Therefore $\beta^*_\xi(f+g) \le \beta_{\tau'}(f+g)$. By Lemma 4.4.13 we have that $\beta_{\tau'}(f) \le \beta_{\tau_f}(f)$ (in fact equality holds), and similarly for g. But $\beta_{\tau'}$ is additive by Theorem 4.2.29, so

$$\beta^*_\xi(f+g) \le \beta_{\tau'}(f+g) \lesssim \max\{\beta_{\tau'}(f),\beta_{\tau'}(g)\} \le \max\{\beta_{\tau_f}(f),\beta_{\tau_g}(g)\} =$$

$$\max\{\beta^*_\xi(f),\beta^*_\xi(g)\}.$$

$\square$

Remark 4.4.15. One can easily deduce from Theorem 4.4.14 that $\beta^*_\xi(f\cdot g) \lesssim \max\{\beta^*_\xi(f),\beta^*_\xi(g)\}$ for every $\xi < \omega_1$ whenever f and g are bounded Baire class ξ functions, and similarly for γ^*_ξ. Again, as in the case of β and γ, the situation is unclear for unbounded functions.

Question 4.4.16. *Let $1 \leq \xi < \omega_1$. Are the ranks β^*_ξ and γ^*_ξ essentially multiplicative?*

Theorem 4.4.17. *If f is a Baire class ξ function then*

$$\alpha^*_\xi(f) \leq \alpha_\xi(f) \leq 2\alpha^*_\xi(f), \text{ hence } \alpha^*_\xi(f) \approx \alpha_\xi(f).$$

Proof. For $\xi = 1$ the claim is an easy consequence of the definition of the two ranks and Corollary 4.2.14. From now on, we suppose that $\xi \geq 2$.

For the first inequality, for every pair of rationals $p < q$ pick a sequence $(F^\zeta_{p,q})_{\zeta<\alpha_\xi(f)} \subseteq \mathbf{\Pi}^0_\xi(X)$, whose transfinite difference separates the level sets $\{f \leq p\}$ and $\{f \geq q\}$.

Every $\mathbf{\Pi}^0_\xi(X)$ set is the intersection of countably many $\mathbf{\Delta}^0_\xi$ sets, hence $F^\zeta_{p,q} = \bigcap_n H^\zeta_{p,q,n}$, with $H^\zeta_{p,q,n} \in \mathbf{\Delta}^0_\xi$. By Kuratowski's theorem [40, 22.18], there is a finer Polish topology $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$, for which $H^\zeta_{p,q,n} \in \mathbf{\Delta}^0_1(\tau')$ for every p, q, n and $\zeta < \alpha_\xi(f)$, hence $F^\zeta_{p,q} \in \mathbf{\Pi}^0_1(\tau')$.

This means that the level sets of f can be separated by transfinite differences of closed sets with respect to τ', hence they can be separated by sets in $\mathbf{\Delta}^0_2(\tau')$. Then it is easy to see that for every $c \in \mathbb{R}$ the level sets $\{f \leq c\}$ and $\{f \geq c\}$ are countable intersections of $\mathbf{\Delta}^0_2(\tau')$ sets, hence they are $\mathbf{\Pi}^0_2(\tau')$ sets, proving that $f \in \mathcal{B}_1(\tau')$. Moreover, $\alpha_{1,\tau'}(f) \leq \alpha_\xi(f)$ easily follows from the construction (here $\alpha_{1,\tau'}$ is the rank α_1 with respect to τ'). And by Corollary 4.2.14 we have $\alpha^*_\xi \leq \alpha_{\tau'}(f) \leq \alpha_{1,\tau'}(f) \leq \alpha_\xi(f)$, proving the first inequality of the theorem.

For the second inequality, take a topology τ' with $\alpha_{\tau'}(f) = \alpha^*_\xi(f)$. Again, by Corollary 4.2.14, we have $\alpha_{1,\tau'}(f) \leq 2\alpha_{\tau'}(f) = 2\alpha^*_\xi(f)$.

It remains to prove that $\alpha_\xi(f) \leq \alpha_{1,\tau'}(f)$. A τ'-closed set is $\mathbf{\Pi}^0_\xi$ with respect to τ. Therefore, if $(F_\eta)_{\eta<\zeta}$ is a decreasing continuous sequence of τ'-closed sets whose transfinite difference separates $\{f \leq p\}$ and $\{f \geq q\}$ then the same sequence is a decreasing continuous sequence of sets from $\mathbf{\Pi}^0_\xi(\tau)$, proving $\alpha_\xi(f) \leq \alpha_{1,\tau'}(f)$. □

Corollary 4.4.18. *α_ξ and α^*_ξ are essentially linear for bounded functions for every ξ.*

Proof. $\alpha_\xi \approx \alpha^*_\xi$ by the previous theorem, $\alpha^*_\xi \approx \beta^*_\xi$ for bounded functions by Corollary 4.4.7, and β^*_ξ is essentially linear by Theorem 4.4.14. □

From Corollary 4.2.39 we can obtain the appropriate statement for the ranks $\alpha^*_\xi, \beta^*_\xi$ and γ^*_ξ.

Proposition 4.4.19. *If $f = \sum_{i=1}^n c_i \chi_{A_i}$, where the A_i's are disjoint $\mathbf{\Delta}^0_{\xi+1}$ sets covering X and the c_i's are distinct then*

$$\alpha^*_\xi(f) \approx \max_i\{\alpha^*_\xi(\chi_{A_i})\},$$

and similarly for β^*_ξ and γ^*_ξ.

Proof. The additivity of α^*_ξ implies $\alpha^*_\xi(f) \lesssim \max_i\{\alpha^*_\xi(\chi_{A_i})\}$. For the other inequality let τ' be a topology for which f is Baire class 1. Then the characteristic functions χ_{A_i} are also Baire class 1, and hence by Corollary 4.2.39 we obtain $\alpha_{\tau'}(f) \approx \max_i\{\alpha_{\tau'}(\chi_{A_i})\}$. But by the definition of α^*_ξ for every such topology $\alpha^*_\xi(\chi_{A_i}) \leq \alpha_{\tau'}(\chi_{A_i})$, therefore $\max_i\{\alpha^*_\xi(\chi_{A_i})\} \leq \max_i\{\alpha_{\tau'}(\chi_{A_i})\} \approx \alpha_{\tau'}(f)$. Then choosing τ' such that $\alpha_{\tau'}(f) = \alpha^*_\xi(f)$ the proof is complete. □

Theorem 4.4.20. *The ranks α^*_ξ, β^*_ξ and γ^*_ξ are unbounded in ω_1. Moreover, for every non-empty perfect set $P \subseteq X$ and ordinal $\zeta < \omega_1$ there exists a characteristic function $\chi_A \in \mathcal{B}_\xi(X)$ with $A \subseteq P$ such that $\alpha^*_\xi(\chi_A), \beta^*_\xi(\chi_A), \gamma^*_\xi(\chi_A) \geq \zeta$.*

Proof. In order to prove the theorem, by Corollary 4.4.6 it suffices to prove the statement for α^*_ξ. Moreover, instead of $\alpha^*_\xi(\chi_A) \geq \zeta$ it suffices to obtain $\alpha^*_\xi(\chi_A) \gtrsim \zeta$. And this is clear from Theorem 4.4.17 and Corollary 4.3.4. □

Proposition 4.4.21. *If f_n, f are Baire class ξ functions and $f_n \to f$ uniformly then $\beta^*_\xi(f) \leq \sup_n \beta^*_\xi(f_n)$.*

Proof. For every n let $\tau_n \in T_{f_n,\xi}$ with $\beta_{\tau_n}(f_n) = \beta^*_\xi(f_n)$. Using Lemma 4.4.12, let τ' be their common refinement satisfying $\tau' \subseteq \mathbf{\Sigma}^0_\xi(\tau)$, where τ is the original topology. Note that $f_n \in \mathcal{B}_1(\tau')$ for every n, and the Baire class 1 functions are closed under uniform limits [40, 24.4], hence $\tau' \in T_{f,\xi}$. Then by Proposition 4.2.33 and Lemma 4.4.13 we have

$$\beta^*_\xi(f) \leq \beta_{\tau'}(f) \leq \sup_n \beta_{\tau'}(f_n) \leq \sup_n \beta_{\tau_n}(f_n) = \sup_n \beta^*_\xi(f_n).$$

□

Proposition 4.4.22. *If f_n, f are Baire class ξ functions and $f_n \to f$ uniformly then $\alpha^*_\xi(f) \lesssim \sup_n \alpha^*_\xi(f_n)$ and $\gamma^*_\xi(f) \lesssim \sup_n \gamma^*_\xi(f_n)$.*

Proof. Repeat the previous argument but apply Proposition 4.2.42 and Proposition 4.2.34 instead of Proposition 4.2.33. □

Hence we were able to prove most of the known properties of the ranks on the Baire class 1 functions for α^*_ξ, β^*_ξ and γ^*_ξ. All three ranks are translation invariant and unbounded in ω_1. The ranks β^*_ξ and γ^*_ξ are essentially linear, while α^*_ξ is not. The ranks $\alpha^*_\xi, \beta^*_\xi$ and γ^*_ξ behave nicely under uniform limits. This may well be considered as an affirmative answer to the (slightly vague) Question 4.0.1. Moreover, we have the following.

Corollary 4.4.23. *The rank β^*_ξ (or γ^*_ξ) provides an affirmative answer to Question 4.0.2.*

Proof. The proofs of the requirements listed in the question can be found in

- Theorem 4.4.20,
- Theorem 4.4.9,

- Theorem 4.4.14,
- Theorem 4.4.10 (note that $1 + \eta \lesssim \eta$ for every η),

respectively. □

Then, by considering the proof of [22, Theorem 6.2] and replacing the class of Borel functions by $\mathcal{B}_\xi$, the Borel class by the rank β_ξ^* and the functions χ_{B_α} by functions supported in P_α with β_ξ^* rank at least α we obtain the following.

Corollary 4.4.24. *For every $2 \leq \xi < \omega_1$ the solvability cardinal $sc(\mathcal{B}_\xi) \geq \omega_2$, hence under the Continuum Hypothesis $sc(\mathcal{B}_\xi) = \omega_2 = (2^\omega)^+$.*

4.5 Uniqueness of the ranks

As we have seen, the natural unbounded ranks defined on the Baire class ξ functions essentially coincide on the bounded functions. Now we will formulate a general theorem which states that if a rank on the bounded functions has certain natural properties then it must agree with the ranks defined above. Because of some not completely clear technical difficulties we only work out the details in the Baire class 1 case.

The main reason why we treat this result separately and did not use it to prove that the ranks considered so far all agree for bounded functions is the following. So far, formally, a rank was simply a map defined on a set of functions. Now we slightly modify this concept: in this section a rank will be a family of maps $\rho = \{\rho^{(X,\tau)}\}_{(X,\tau)\text{ Polish}}$, where $\rho^{(X,\tau)}$ is a rank on the Baire class 1 functions defined on the Polish space (X, τ). However, since there is no danger of confusion, we will abuse notation and will simply continue to use ρ. Notice that the ranks α, β and γ can naturally be viewed this way.

Theorem 4.5.1. *Let ρ be a rank on the bounded Baire class 1 functions. Suppose that ρ has the following properties for every $A \in \mathbf{\Delta}_2^0$ and Baire class 1 functions f and f_n:*

(1) $\rho(\chi_A) \approx \alpha_1(A, A^c)$ ($\approx \alpha(A, A^c) \approx \alpha(\chi_A) \approx \beta(\chi_A) \approx \gamma(\chi_A)$, that is, the rank of A is essentially its complexity in the difference hierarchy),

(2) ρ is essentially linear,

(3) if $f_n \to f$ uniformly then $\rho(f) \lesssim \sup_n \rho(f_n)$,

(4) if $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $\rho(h \circ f) \lesssim \rho(f)$,

(5) if f is defined on the Polish space X and $Y \subset X$ is Polish (or equivalently, $\mathbf{\Pi}_2^0(X)$, see e.g. [40, 3.11]) then $\rho(f|_Y) \lesssim \rho(f)$.

Then $\rho \approx \alpha$ for bounded Baire class 1 functions.

Property (5) is probably the most ad hoc among the conditions, however it is easy to see that it holds for ranks α, β and γ:

Lemma 4.5.2. *Let X, Y be Polish spaces with $Y \subset X$ and f be a bounded Baire class 1 function on X. Then $\alpha(f|_Y) \lesssim \alpha(f)$, and hence similarly for β and γ.*

Proof. Using Corollary 4.2.14, it is enough to prove the lemma for α_1. By the definition of the rank α_1, if $p < q$ are rational numbers then there exists a $\mathbf{\Delta}^0_2(X)$ set A such that $\alpha_1(A, A^c) \leq \alpha_1(f)$ and A separates $\{f \leq p\}$ and $\{f \geq q\}$. Clearly, $A \cap Y$ separates the sets $\{f|_Y \leq p\}$ and $\{f|_Y \geq q\}$. So it is enough to show that $\alpha_{1,Y}(A \cap Y, A^c \cap Y) \leq \alpha_1(A, A^c)$.

Now, there exists a sequence of closed sets $(F_\eta)_{\eta < \alpha_1(A,A^c)}$ such that

$$A = \bigcup_{\substack{\eta < \alpha_1(A,A^c) \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}).$$

But the sets $(F_\eta \cap Y)_{\eta < \alpha_1(A,A^c)}$ witness that $\alpha_{1,Y}(A \cap Y, A^c \cap Y) \leq \alpha_1(A, A^c)$, so we are done. □

Proof of Theorem 4.5.1. We split the proof of the theorem into two easy lemmas.

Lemma 4.5.3. *If $f = \sum_{i=1}^n c_i \chi_{A_i}$ where the A_i's are disjoint $\mathbf{\Delta}^0_2$ sets covering the underlying space X and the c_i's are distinct then $\rho(f) \approx \alpha(f)$.*

Proof. By the essential linearity of ρ clearly

$$\rho(f) \lesssim \max_i \rho(\chi_{A_i}).$$

Now let $0 \leq j \leq n$ be fixed and $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz such that $h(c_i) = 0$ for $i \neq j$ and $h(c_j) = 1$. Then

$$\rho(\chi_{A_j}) = \rho(h \circ f) \lesssim \rho(f)$$

by Property (4), so we have that

$$\rho(f) \approx \max_i \rho(\chi_{A_i}).$$

Using Corollary 4.2.39 and Property (1) we obtain that α and ρ essentially agree on step functions. □

Now let f be an arbitrary bounded Baire class 1 function. Then by Lemma 4.2.40 and Proposition 4.2.42 there exists a sequence of step functions f_n converging uniformly to f such that $\alpha(f) \approx \sup_n \alpha(f_n)$. Hence, by Property (3) and the previous lemma,

$$\rho(f) \lesssim \sup_n \rho(f_n) \approx \sup_n \alpha(f_n) \approx \alpha(f).$$

Hence, interchanging the role of α and ρ in the above argument, in order to prove $\rho(f) \approx \alpha(f)$ it is enough to construct a sequence f_n of step functions converging uniformly to f such that

$$\sup_n \rho(f_n) \lesssim \rho(f). \tag{4.5.1}$$

The construction goes similarly to that of Lemma 4.2.40, but we need an additional step.

Lemma 4.5.4. *Suppose that f is a bounded Baire class 1 function on the Polish space X and $p, q \in \mathbb{R}$ with $p < q$. Then there exists a set $H \in \mathbf{\Delta}^0_2(X)$ such that $\rho(\chi_H) \lesssim \rho(f)$ and H separates the sets $\{f \leq p\}$ and $\{f \geq q\}$.*

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz such that $h|_{(-\infty,p]} \equiv 0$ and $h|_{[q,\infty)} \equiv 1$. Let $f_1 = h \circ f$, Property (4) ensures that

$$\rho(f_1) \lesssim \rho(f). \tag{4.5.2}$$

Let $Y = \{f \leq p\} \cup \{f \geq q\}$ and $f_2 = f_1|_Y$. Clearly, f_2 is a step function on the Polish space Y (note that Y is $\mathbf{\Pi}^0_2(X)$), hence by the previous lemma and Property (5) we obtain

$$\alpha(f_2) \approx \rho(f_2) \lesssim \rho(f_1). \tag{4.5.3}$$

In particular, $\{f_2 \leq 0\}$ and $\{f_2 \geq 1\}$ can be separated by a $\mathbf{\Delta}^0_2(Y)$ set H' such that

$$H' = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F'_\eta \setminus F'_{\eta+1})$$

for some $F'_\eta \in \mathbf{\Pi}^0_1(Y)$ and

$$\lambda \lesssim \alpha(f_2), \tag{4.5.4}$$

using Corollary 4.2.14.

Now let F_η be the closure of F'_η in X and

$$H = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}).$$

Then H is a $\mathbf{\Delta}^0_2(X)$ set, and by Property (1), Corollary 4.2.14, (4.5.4), (4.5.3) and (4.5.2) we obtain

$$\rho(\chi_H) \approx \alpha(\chi_H) \leq \lambda \lesssim \alpha(f_2) \approx \rho(f_2) \lesssim \rho(f_1) \lesssim \rho(f).$$

Moreover,

$$H \cap Y = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F_\eta \cap F^c_{\eta+1} \cap Y) = \bigcup_{\substack{\eta<\lambda \\ \eta \text{ even}}} (F'_\eta \cap F'^c_{\eta+1} \cap Y) = H' \cap Y.$$

Since H' separates $\{f_2 \leq 0\}$ and $\{f_2 \geq 1\}$, and it is easy to see that $\{f \leq p\} \subset \{f_2 \leq 0\} \subset Y$ and analogously for $\{f \geq q\}$, we obtain that H separates $\{f \leq p\}$ and $\{f \geq q\}$, which completes the proof. $\square$

Now we complete the proof by constructing a sequence f_n converging uniformly to f and satisfying (4.5.1). We basically repeat the proof of Lemma 4.2.40. Let $p_{n,k} = k/2^n$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $\inf(f) \leq p_{n,k} \leq \sup(f)$. By the boundedness of f there are just finitely many $p_{n,k}$'s. The level sets $\{f \leq p_{n,k}\}$ and $\{f \geq p_{n,k+1}\}$ are disjoint $\mathbf{\Pi}^0_2$ sets, hence by the previous lemma they can be separated by a $H_{n,k} \in \mathbf{\Delta}^0_2$ so that $\rho(\chi_{H_{n,k}}) \lesssim \rho(f)$. Set

$$f_n = \sum_k p_{n,k} \cdot (\chi_{H_{n,k+1}} - \chi_{H_{n,k}}).$$

Clearly, $f_n \to f$ uniformly. Now, for every n

$$\rho(f_n) = \rho\left(\sum_k p_{n,k} \cdot (\chi_{H_{n,k+1}} - \chi_{H_{n,k}})\right) \lesssim \max_k \rho(\chi_{H_{n,k}}) \lesssim \rho(f)$$

by the essential linearity of ρ, which finishes the proof of the theorem. □

It is not hard to see that if the range of our functions is 2^ω instead of $\mathbb{R}$ (or any other zero dimensional linearly ordered Polish space) then we can drop Property (5) in Theorem 4.5.1.

Question 4.5.5. *Does there exist a rank ρ with Properties $(1) - (4)$, such that $\rho \not\approx \alpha$?*

Now we very briefly discuss the Baire class ξ case. It is not hard to check that if the family of ranks is defined not only on functions on the Polish spaces, but also on functions on all subsets (or just Borel or $\mathbf{\Pi}^0_{\xi+1}$ subsets) of Polish spaces, and Property (5) is modified accordingly, then a result analogous to Theorem 4.5.1 holds. However, the following question, where the ranks are only defined on functions on the Polish spaces is more natural.

Question 4.5.6. *Let ρ be rank on the bounded Baire class ξ functions (defined on Polish spaces). Suppose that ρ has the following properties:*

(1) if $A \in \mathbf{\Delta}^0_{\xi+1}(X)$ then $\rho(\chi_A) \approx \alpha_\xi(\chi_A)$,

(2) ρ is essentially linear,

(3) if $f_n \to f$ uniformly then $\rho(f) \lesssim \sup_n \rho(f_n)$,

(4) if $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $\rho(h \circ f) \lesssim \rho(f)$,

(5) if $H \in \mathbf{\Pi}^0_2(X)$ then $\rho(f|_H) \lesssim \rho(f)$.

Does this imply that $\rho \approx \alpha$ for bounded Baire class ξ functions?

4.6 Open problems

In this last section we collect the open problems of the chapter.

Throughout the chapter we almost always considered only the relations $\approx$ and $\lesssim$. It would be interesting to know which statements remain true using $=$ and $\leq$ instead.

Question 4.6.1. *Let ρ and ρ' be two of the ranks defined in this chapter for which $\rho \lesssim \rho'$ holds. Is it true that $\rho \leq \rho'$?*

We have shown in Theorem 4.3.8 that if $1 \leq \xi < \omega_1$ and f is a *characteristic* Baire class ξ function then the linearised separation rank $\alpha'_\xi(f) \leq 2$.

Question 4.6.2. *Is the linearised separation rank α'_ξ unbounded in ω_1 for the Baire class ξ functions?*

Actually, we do not even know the answer in the case of $\xi = 1$.

The following question is very closely related to this.

Question 4.6.3. *Let $1 \leq \xi < \omega_1$ and let f_n and f be Baire class ξ functions such that $f_n \to f$ uniformly. Does this imply that $\alpha'_\xi(f) \lesssim \sup_n \alpha'_\xi(f_n)$?*

As mentioned above, an affirmative answer to this question would provide a negative answer to the previous one.

Recall that a rank ρ is *essentially multiplicative* if $\rho(f \cdot g) \lesssim \max\{\rho(f), \rho(g)\}$ for every f and g. Remarks 4.2.31 and 4.4.15 indicate that the ranks β, γ, β^*_ξ and γ^*_ξ are essentially multiplicative on the *bounded* functions from the appropriate Baire classes.

Question 4.6.4. *Let $1 \leq \xi < \omega_1$. Are the ranks β, γ, β^*_ξ and γ^*_ξ essentially multiplicative?*

We have shown in Theorem 4.3.12 that the limit ranks are bounded by ω, but do not know whether this is optimal.

Question 4.6.5. *Is there an $n \in \omega$ such that $\overline{\gamma}_2 \leq n$? If yes, which is the smallest such n?*

We have seen that for every $1 \leq \xi < \omega_1$ we have $\beta^*_\xi \approx \gamma^*_\xi$ on the bounded Baire class ξ functions (even on non-compact Polish spaces), but $\alpha^*_\xi \not\approx \beta^*_\xi$ for arbitrary Baire class ξ functions. So the following question is natural.

Question 4.6.6. *Let $1 \leq \xi < \omega_1$. Does $\beta^*_\xi \approx \gamma^*_\xi$ hold for arbitrary Baire class ξ functions?*

We believe that an affirmative answer might help extend Theorem 4.5.1 to the unbounded case.

Our next questions concern the uniqueness of ranks.

Question 4.6.7. *Does there exist a rank ρ with Properties* $(1)-(4)$ *of Theorem 4.5.1 such that $\rho \not\approx \alpha$ on bounded Baire class* 1 *functions?*

Question 4.6.8. *Let ρ be rank on the bounded Baire class ξ functions (defined on Polish spaces). Suppose that ρ has the following properties:*

1. *if $A \in \mathbf{\Delta}^0_{\xi+1}(X)$ then $\rho(\chi_A) \approx \alpha_\xi(\chi_A)$,*
2. *ρ is essentially linear,*
3. *if $f_n \to f$ uniformly then $\rho(f) \lesssim \sup_n \rho(f_n)$,*
4. *if $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function then $\rho(h \circ f) \lesssim \rho(f)$,*
5. *if $H \in \mathbf{\Pi}^0_2(X)$ then $\rho(f|_H) \lesssim \rho(f)$.*

Does this imply that $\rho \approx \alpha$ for bounded Baire class ξ functions?

Question 4.6.9. *The fourth chapter of [42] discusses two more ranks on the bounded Baire class 1 functions that turn out to be essentially equivalent to α, β and γ. Is there a well-behaved generalisation of these theories to the Baire class ξ case?*

Chapter 5

Coanalytic sets and $V = L$

A two-point set is a subset of the plane that intersects every line in exactly two points. Mazurkiewicz showed the existence of a two-point set using transfinite induction. Erdős asked whether a two-point set can be a Borel set. This question is still open.

A. W. Miller proved in [52] that under certain set theoretic assumptions (namely $V = L$, where L denotes Gödel's constructible universe) one can construct a coanalytic two-point set. Miller also proved the consistent existence of a coanalytic MAD family (i. e. a maximal infinite family of infinite subsets of ω such that the intersection of any two elements is finite) and Hamel basis. The author proves the statement only for two-point sets and the proof uses deep set theoretical tools. References to Miller's method appear in several papers ([29], [31], [39] etc.), sometimes omitting the proof. However, the first version of the method was published by Erdős, Kunen and Mauldin [28].

Our aim in this chapter is to make precise and prove a "black box" condition which could easily be applied without the set theoretical machinery.

Let us remark here that in all of the above mentioned cases, except of course the two-point set, the class of coanalytic sets is best possible, since it is known that there is no analytic

(1) MAD family,

(2) Hamel basis,

(3) C^1-small set (that is, an uncountable subset of the plain that intersects every C^1 curve in countably many points).

(1) is a classical result of Mathias [49] and for the proof of (3) see [33]. (2) can be shown with an easy computation. Moreover, assuming projective determinacy one can show that there is no projective Hamel basis or C^1-small set. It is also an interesting fact that an analytic two-point set is automatically Borel.

Now, in order to formulate our results first we define Turing reducibility. Throughout the chapter M will stand for $\mathbb{R}^n$, 2^ω, $\mathcal{P}(\omega)$ or ω^ω.

Definition 5.0.1. Suppose that $x, y \in M$. We say that x *is Turing reducible to* y if there exists a Turing machine such that if the digits of y are written on the second tape and an input $n \in \mathbb{N}$ is written on the first tape then it stops in finitely many steps and outputs the nth digit of x on the first tape. This relation is denoted by $x \leq_T y$. Let us say that $A \subset M$ is *cofinal in the Turing degrees*, if for every $x \in M$ there exists a $y \in A$ such that $x \leq_T y$.

Roughly speaking, the theorem will state that if given a transfinite induction that picks a real x_α at each step α, the set of possible choices (described by the set F below) is nice enough and cofinal in the Turing degrees then the induction can be realised so that it produces a coanalytic set. In most cases there will be an extra requirement that x_α has to be picked from a given set H_α. For example, in the construction of the two-point set H_α is the αth line. Instead of the sets H_α we will use a parametrisation where H_α will be coded by p_α and typically the codes will range over $\mathbb{R}$. The set of the codes will be denoted by B.

Notation. For a set H the set of countable sequences of elements of H is denoted by $H^{\leq\omega}$. Note that if M is a Polish space then there is a natural Polish structure on $M^{\leq\omega}$.

Definition 5.0.2. Let $F \subset M^{\leq\omega} \times B \times M$, and $X \subset M$. We say that X *is compatible with* F if there exist enumerations $B = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ a sequence $A_\alpha \in M^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq \omega$ such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$ holds.

This definition is basically describing that in each step of the transfinite induction we pick an element from a set $F_{(A_\alpha, p_\alpha)}$ which depends on the set of the previous choices A_α and the αth parameter p_α.

Theorem 5.0.3. *($V = L$) Let B be an uncountable Borel subset of an arbitrary Polish space. Suppose that $F \subset M^{\leq\omega} \times B \times M$ is a coanalytic set and for all $p \in B$, $A \in M^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the Turing degrees. Then there exists a coanalytic set X that is compatible with F.*

In fact we will prove a much stronger theorem (Theorem 5.2.4) which we call the Main Theorem of this chapter. However, all the classical applications are using Theorem 5.0.3 and it will be an easy consequence of the Main Theorem (see Section 5.3). We would like to emphasise one of our further results from Section 5.3.

Theorem 5.0.4. *($V = L$) Suppose that $G \subset \mathbb{R} \times \mathbb{R}^n$ is a Borel set and for every countable $A \subset \mathbb{R}$ the complement of the set $\cup_{p \in A} G_p$ is cofinal in the Turing degrees. Then there exists an uncountable coanalytic set $X \subset \mathbb{R}^n$ that intersects for every $p \in \mathbb{R}$ the section G_p in a countable set.*

The chapter is organised as follows: in Section 5.1 we summarise the most important facts used for the proof and Section 5.2 contains the proof of the Main Theorem. In Section 5.3 we prove several generalisations, a partial converse and we obtain the existence of a coanalytic Hamel basis (which slightly differs from the other applications).

In Section 5.4 we present the applications of our theorem and finally, in Section 5.5 we mention some open problems.

Unfortunately, the proof of the main result of this chapter uses quite a lot of known facts about the model L and other theorems of computability theory. The full definition and explanation of these concepts would exceed the framework of this thesis. Therefore, as opposed to the previous chapters, the proof of the main result will heavily build on basic theorems about the model L and computability theory (see [15], [54]). The reader only interested in how to apply the method developed in this chapter may now proceed to Section 5.4, which is not building on the technicalities of Sections 5.1, 5.2 and 5.3.

5.1 Preliminaries

In this chapter we will follow the notation of [54]. We identify ω^ω, $(\omega^\omega)^{\leq\omega}$, 2^ω, $\mathbb{R}^{\leq\omega}$, $\mathcal{P}(\omega)$ and their finite products, since there are recursive Borel-isomorphisms between them ([54, 3I.4.Theorem]). A "real" is an element of one of these spaces. For convenience we will use ω^ω in most cases. If $A \in (\omega^\omega)^{\leq\omega}$ and $n \in \omega$, let us denote the nth element of A (as a sequence) with $A(n)$.

If t is a real, let us denote the classes of the arithmetic and projective hierarchy recursive in t with $\Sigma^i_j(t)$, $\Delta^i_j(t)$ and $\Pi^i_j(t)$ $(i = 0, 1,\ j \in \omega)$. Thus for example the set of coanalytic subsets of ω^ω equals $\bigcup_{t\in\omega^\omega} \Pi^1_1(t)$. For $t \equiv 0$ we will write Σ^i_j instead of $\Sigma^i_j(t)$ etc.

The theorems we will use can be found in [55] and [15], but we recall the most important facts. Let us denote the set of self-constructible reals, i.e. $\{x \in \omega^\omega : x \in L_{\omega_1^x}\}$ with $\mathcal{S}$, where ω_1^x is the first ordinal not recursive in x and L_α is the αth level of Gödel's constructible universe, L. Let $<_L$ be the standard well ordering of L.

Theorem 5.1.1. *([41, Theorem (2A-1)]) $\mathcal{S}$ is a Π^1_1 set.*

For reals x, y let us denote by $x \leq_h y$ that x is hyperarithmetic in y or equivalently $x \in \Delta^1_1(y)$ (see [55] or [51, Corollary 27.4]). If A is a set, $L_\alpha[A]$ denotes the αth level of the universe constructed from A, that is, in the initial step we start from $\emptyset$ and A.

Theorem 5.1.2. *([55, Part A, Chapter II, 7]) $x \leq_h y$ is a Π^1_1 relation and for arbitrary reals it is equivalent to $x \in L_{\omega_1^y}[y]$. Moreover, $x \leq_h y$ implies $\omega_1^x \leq \omega_1^y$.*

We will use the following form of the Spector-Gandy Theorem:

Theorem 5.1.3. *([51, Corollary 29.3]) Let $A \subset (\omega^\omega)^2$ be a $\Pi^1_1(t)$ subset of $(\omega^\omega)^2$. Then the set*

$$(\exists y \leq_h x)((x, y) \in A)$$

is also $\Pi^1_1(t)$.

In [10] the authors work with a very useful alternative form. We call a formula in the language of set theory Σ_1 if it has just one unbounded quantifier and that is existential. In case all the quantifiers are bounded, we call it Δ_0.

Theorem 5.1.4. *A set A is $\Pi^1_1(t)$ if and only if there exists a Σ_1 formula θ such that*

$$x \in A \iff L_{\omega_1^{(x,t)}}[x,t] \models \theta(x,t).$$

Definition 5.1.5. We call a set $X \subset \omega^\omega$ *cofinal in the hyperdegrees* if for every $y \in \omega^\omega$ there exists an $x \in X$ such that $y \leq_h x$.

Furthermore, in [10] one can find the following lemma.

Lemma 5.1.6. *($V = L$) Let $t \in \omega^\omega$ be arbitrary. A $\Pi^1_1(t)$ set X is cofinal in the hyperdegrees if and only if $X \cap \mathcal{S}$ is cofinal in the hyperdegrees.*

5.2 The main theorem

First we will prove a rather technical lemma.

Lemma 5.2.1. *Suppose that $\theta(s,p,q)$ is a Σ_1 formula of set theory. Then there exists a Σ_1 formula $\theta'(s,p)$ such that for every limit ordinal $\alpha > \omega$*

$$L_\alpha \models ((\forall q <_L p)(\theta(s,p,q)) \iff \theta'(s,p)).$$

Proof. By [15, 3.5 Lemma, p. 75] there exists a Σ_1 formula $\zeta(x,y)$ such that for arbitrary limit ordinal $\alpha > \omega$ and $x, y \in L_\alpha$

$$L_\alpha \models (\zeta(x,y) \iff y = \{t : t <_L x\}).$$

Notice that if $\alpha > \omega$ is a limit ordinal and $x \in L_\alpha$ then $\{t : t <_L x\} \in L_\alpha$. Let

$$\theta''(s,p) = (\exists y)(\zeta(p,y) \wedge (\forall q \in y)(\theta(s,p,q))).$$

Now, since θ'' contains only existential and bounded quantifiers, using the well-known trick there exists a Σ_1 formula $\theta'(s,p)$ such that for every limit ordinal $\alpha > \omega$

$$L_\alpha \models (\theta''(s,p) \iff \theta'(s,p)).$$

□

In the following lemma we will select a single well-ordering of ω of type α for every countable ordinal α in a "nice" way. The selection will be done by a formula $\phi(z,x)$ that intuitively means that x "knows" that z is a canonical well-ordering. Let $z \subset \omega^2$ and define $<_z$ as the relation $m <_z n \iff (m,n) \in z$. Let us use the notation $dom(<_z)$ for the set $\{n \in \omega : (\exists m \in \omega)((m,n) \in z)\}$. For $z, z' \in \mathcal{P}(\omega^2)$ we say that $<_z \cong <_{z'}$ if there exists a bijection $f : dom(<_z) \to dom(<_{z'})$ such that

$$(\forall m, n \in dom(<_z))(m <_z n \iff f(m) <_{z'} f(n)).$$

Now if $<_z$ is an ordering and $n \in \omega$ let us denote by $<_z |_{<_z n}$ the ordering obtained by restricting $<_z$ to the set $\{m \in \omega : m <_z n\}$.

Lemma 5.2.2. *($V = L$) There exists a formula $\phi(z,x)$ defining a Π^1_1 subset of $\mathcal{P}(\omega^2) \times \omega^\omega$ with the following properties*

(1) if $s \subset \omega^2$ and $<_s$ is a well-ordering then there exists a unique z such that $<_z \cong <_s$, $(\exists x \in \omega^\omega)\phi(z,x)$ and $dom(<_z)$ is a natural number or ω

(2) if $y \in \mathcal{S}$, $x \leq_h y$ and $\phi(z,x)$ then $\phi(z,y)$

(3) if $\phi(z,x)$ then $z \leq_h x$ and $x \in \mathcal{S}$

(4) if $\phi(z,x)$ and $n \in \omega$ is arbitrary then there exists a unique pair $g_n, y_n \in L_{\omega_1^x}$ such that $\phi(y_n,x)$ and $g_n \subset \omega^2$ is an isomorphism between $<_z |_{<_z n}$ and $<_{y_n}$.

Proof. First let us denote by $\psi(z,h,\alpha)$ the conjunction of the following three formulas:

- h is a function, $dom(h) = \alpha$ is an ordinal, $ran(h) = dom(<_z)$,
- $(\forall \beta, \beta' \in \alpha)(\beta \in \beta' \iff h(\beta) <_z h(\beta'))$,
- $dom(<_z)$ is a natural number or ω.

So $\psi(z,h,\alpha)$ says that h is an isomorphism between α and $<_z$. Notice that ψ is a Δ_0 formula (see [15], Section I). Hence for limit ordinals $\beta > \omega$ if $z, h, \alpha \in L_\beta$ then $L \models \psi(z,h,\alpha) \iff L_\beta \models \psi(z,h,\alpha)$.

Let us define $\phi(z,x)$ as follows:

$$\phi(z,x) \iff x \in \mathcal{S} \wedge z \leq_h x \wedge$$

$$L_{\omega_1^x} \models (\exists h \exists \alpha)\big((\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big).$$

First, we will prove that $\phi(z,x)$ defines a Π^1_1 set. The formula

$$(\exists h \exists \alpha)\big((\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big)$$

by Lemma 5.2.1 is equivalent to a Σ_1 formula, say $\zeta(z)$, in L_β if β is a limit ordinal and $\beta > \omega$. Notice that $z \leq_h x$ implies $(x,z) \leq_h x$ so $\omega_1^{(x,z)} \leq \omega_1^x$ by Theorem 5.1.2. Moreover, from $(x,z) \leq_h x$ and by Theorem 5.1.2 we have that $(x,z) \in L_{\omega_1^x}[x]$. Additionally, $x \in \mathcal{S}$ so $L_{\omega_1^x} = L_{\omega_1^x}[x]$. Thus $(x,z) \in L_{\omega_1^x}$ and the equality $L_{\omega_1^{(x,z)}}[x,z] = L_{\omega_1^x}$ holds. Therefore

$$L_{\omega_1^x} \models (\exists h \exists \alpha)\big((\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big)$$

$$\iff$$

$$L_{\omega_1^x} \models \zeta(z)$$

$$\iff$$

$$L_{\omega_1^{(x,z)}}[x,z] \models \zeta(z).$$

By Theorems 5.1.1 and 5.1.2 it is clear that $(x \in \mathcal{S}) \wedge (z \leq_h x)$ defines a Π^1_1 set. Now we can prove that the set $\{(x,z) : L_{\omega_1^{(x,z)}}[x,z] \models \zeta(z)\}$ is also Π^1_1 using Theorem 5.1.4 with $t = 0$ and replacing x by (x,z). Thus ϕ defines a Π^1_1 set.

Now we will prove that $\phi(z,x)$ has the required properties.

(1) Let $s \subset \omega^2$ be an arbitrary well-ordering. Then $<_s$ is isomorphic to some ordinal α. There exists a $<_L$ minimal pair (z,h) such that h is an isomorphism between $<_z$ and α and $\mathrm{dom}(<_z)$ is a natural number or ω. Therefore

$$L \models (\exists h \exists \alpha)\big((\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big).$$

Notice that if $\xi(s)$ is a Δ_0 formula, β is a limit ordinal such that $s \in L_\beta$ and $L \models \xi(s)$ then $L_\beta \models \xi(s)$. Therefore automatically $L_\beta \models (\exists r)(\xi(r))$. Considering this one can conclude that

$$L_{\omega_1^x} \models (\exists h \exists \alpha)\big((\psi(z,h,\alpha) \wedge (\forall (z',h') <_L (z,h))(\neg\psi(z',h',\alpha))\big)$$

holds if $(z,h) \in L_{\omega_1^x}$. $\mathcal{S}$ is cofinal in the hyperdegrees (Lemma 5.1.6) hence there exists an $x \in \mathcal{S}$ such that $(z,h) \in L_{\omega_1^x}$. So for such an x we have $\phi(z,x)$.

(2) To prove the second claim just observe that Σ_1 formulas are upward absolute for transitive sets and notice that $x \leq_h y$ implies that $L_{\omega_1^x} \subset L_{\omega_1^y}$.

(3) Obvious from the definition of ϕ.

(4) Let $x \in \omega^\omega$, $z \subset \omega^2$, $n \in \omega$ and assume that $\phi(z,x)$ holds. Clearly there exists a unique ordinal $\beta < \alpha$ such that $\beta \cong <_z |_{<_z n}$.

First we will prove that there exists a pair $(y'_n, h'_n) \in \mathcal{P}(\omega^2) \times \beta^{dom(<_{y'_n})}$ such that $L_{\omega_1^x} \models \psi(y'_n, h'_n, \beta)$. We know that $L_{\omega_1^x} \models \psi(z,h,\alpha)$ for some $h, \alpha \in L_{\omega_1^x}$ so the same holds in L. The fact that $\psi(z,h,\alpha)$ holds implies that h is an isomorphism between $<_z$ and α, so $h' = h|_\beta$ is an isomorphism between β and $<_z |_{<_z n}$. Obviously, $h' \in L_{\omega_1^x}$, so there exists an ordinal $\gamma < \omega_1^x$ such that $h' \in L_\gamma$.

Let $e : \omega \to ran(h')$ be defined as follows:

$$\langle m,k \rangle \in e \iff (k \in ran(h') \wedge \exists e'(e' : m \leftrightarrow ran(h') \cap k+1)),$$

in other words, there exists a bijection between m and the initial segment of $ran(h')$, or equivalently, $|\{l \in ran(h') : l \leq k\}| = m$. Since the bijections between the finite subsets of ω are already in L_ω, we have that $e \in L_{\gamma+2} \subset L_{\omega_1^x}$. e is clearly a one-to-one function from a finite number or ω onto $ran(h')$.

Now take $\langle k,l \rangle \in y'_n \iff \langle e(k), e(l) \rangle \in z$ and $h'_n = e^{-1} \circ h'$. Then $L \models \psi(y'_n, h'_n, \beta)$ and of course $y'_n, h'_n, \beta \in L_{\omega_1^x}$ hence $L_{\omega_1^x} \models \psi(y'_n, h'_n, \beta)$.

Thus there exists a $<_L$ minimal pair $(y_n, h_n) \in L_{\omega_1^x}$ such that $L_{\omega_1^x} \models \psi(y_n, h_n, \beta)$. Note that the $<_L$ ordering is absolute for L_α and L if $\alpha > \omega$ is a limit ordinal, so

$L_{\omega_1^x} \models$ "(y_n, h_n) is the $<_L$ minimal pair such that $\psi(y_n, h_n, \beta)$". By Theorem 5.1.2, if $y_n \in L_{\omega_1^x}$ then $y_n \leq_h x$. Thus $\phi(y_n, x)$ holds.

Finally recall that $h_n : \beta \to dom(<_{y_n})$ and $h' : \beta \to dom(<_z |_{<_z n})$ are isomorphisms in $L_{\omega_1^x}$. So the function $g_n = h_n \circ (h')^{-1}$ is in $L_{\omega_1^x}$. This is an isomorphism between two well-orderings so this is unique.

□

Let us recall the definition of compatibility.

Definition 5.2.3. Let $F \subset M^{\leq\omega} \times B \times M$, $X \subset M$. We say that X *is compatible with* F if there exist enumerations $B = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ a sequence $A_\alpha \in M^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq \omega$ such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$ holds.

Theorem 5.2.4. *(Main Theorem) ($V = L$) Let $t \in \omega^\omega$. Suppose that $F \subset (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$ is a $\Pi^1_1(t)$ set and for all $p \in \omega^\omega$, $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees. Then there exists a $\Pi^1_1(t)$ set $X \subset \omega^\omega$ that is compatible with F.*

Proof of the Main Theorem. In the first step we will modify the set F. Let us define $F' \subset \mathcal{P}(\omega^2) \times (\omega^\omega)^{\leq\omega} \times (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$ as follows

$(z, A, P, p, x) \in F' \iff$

(1) $\phi(z, x)$ (in particular $x \in \mathcal{S}$)

(2) $A, P, p, t \leq_h x$, $(A, p, x) \in F$

(3) $L_{\omega_1^x} \models \exists g$

   (a) g is a function, $dom(g) \in \omega \cup \{\omega\}$, $ran(g) = P$

   (b) $(\forall n, m \in dom(g))$ $(n <_z m \iff g(n) <_L g(m))$

   (c) $(\forall p' <_L p)(p' \in \omega^\omega \Rightarrow (\exists n \in \omega)(g(n) = p'))$

The role of z is that it will encode the history of the previous choices. $(1) - (2)$ basically ensures that x is complicated enough. The clauses (a) and (b) describe that P is an enumeration in type $\leq \omega$ of the first α reals with respect to $<_L$, where $\alpha = tp(<_z)$. (c) is the formalisation of $L_{\omega_1^x} \models$ "p is the αth real with respect to $<_L$".

Lemma 5.2.2, Theorems 5.1.1 and 5.1.2 guarantee that the (1) and (2) are defining a $\Pi^1_1(t)$ set.

We can prove that (3) defines a Π^1_1 set similarly as we did in Lemma 5.2.2: (a) and (b) are Δ_0 formulas, (c) is Σ_1 by Lemma 5.2.1. So by the well-known technical trick the conjunction is equivalent to a Σ_1 formula. Moreover we know that for arbitrary reals $a \leq_h b \iff a \in L_{\omega_1^b}[b]$ and $a \leq_h b$ implies $\omega_1^a \leq \omega_1^b$. Therefore by (1) and (2)

$$L_{\omega_1^{(z,A,P,p,t,x)}}[z, A, P, p, t, x] = L_{\omega_1^x}$$

and using the Spector-Gandy Theorem (Theorem 5.1.4) we can conclude that F' is a $\Pi^1_1(t)$ set.

Remark 5.2.5. By absoluteness, if $(z, A, P, p, x) \in F'$ then P must be the enumeration of the first α reals given by $<_z$ in L as well. Similarly p must be the αth real with respect to $<_L$ (where $\alpha = tp(<_z)$).

Lemma 5.2.6. *Suppose that* $x \in F'_{(z,A,P,p)}$, $x \leq_h y$ *and* $y \in \mathcal{S} \cap F_{(A,p)}$. *Then* $y \in F'_{(z,A,P,p)}$.

Proof. Let x, y be reals satisfying the conditions above. Now considering the definition of F', the formula $\phi(z, y)$ holds by the second claim of Lemma 5.2.2. Of course, $A, P, p, t \leq_h x$ implies $A, P, p, t \leq_h y$. Finally, $L_{\omega_1^x} \subset L_{\omega_1^y}$, by Theorem 5.1.2, and the formula in (3) that must hold in $L_{\omega_1^y}$ does not depend on x, hence it is also true in $L_{\omega_1^y}$. □

Lemma 5.2.7. *If the section* $F'_{(z,A,P,p)}$ *is non-empty then it is cofinal in the hyperdegrees.*

Proof. Fix an arbitrary $s \in \omega^\omega$ and let $x \in F'_{(z,A,P,p)}$. By the assumptions of the Main Theorem each section $F_{(A,p)}$ is cofinal in the hyperdegrees. Using Lemma 5.1.6 we have that there exists a $y \in F_{(A,p)} \cap \mathcal{S}$ such that $s, x \leq_h y$. Thus by the previous lemma $y \in F'_{(z,A,P,p)}$ and this proves the statement. □

Now we select a real from each non-empty section of F'. Let $F'' \subset F'$ be a $\Pi^1_1(t)$ uniformisation of F', that is, for all $(z, A, P, p) \in proj(F')$ we have $|F''_{(z,A,P,p)}| = 1$ (see [51] or [55] for the relative version of the uniformisation theorem).

There may be elements $(z, A, P, p, x) \in F''$ with "wrong" history, namely $A(n)$ may not be a selected real for some $n \in \omega$. So we have to sort out the appropriate ones.

Let $F''' \subset F''$ be defined as follows:

$(z, A, P, p, x) \in F''' \iff$

(1) $(z, A, P, p, x) \in F''$

(2) $(\forall n \in \omega)(\exists g_n, y_n \leq_h x)$

  (a) $\phi(y_n, x)$

  (b) g_n is an isomorphism between $<_z |_{<_z n}$ and y_n

  (c) if $A_n, P_n \in (\omega^\omega)^{\leq\omega}$ is defined by $A_n(i) = A(g_n(i))$ and similarly $P_n(i) = P(g_n(i))$ then $(y_n, A_n, P_n, P(n), A(n)) \in F''$

By properties of ϕ, for every countable ordinal α we have a canonical enumeration of α. In the definition above (c) ensures that for every $(z, x, A, P, p) \in F'''$ the set A is the canonical enumeration of the previous choices given by the uniformisation of F'.

The clauses $(a), (b)$ are defining a $\Pi^1_1(t)$ set. Now take the map

$$\Psi : (A, P, y_n, g_n, n) \mapsto (y_n, A \circ g_n, P \circ g_n, P(n), A(n)).$$

Observe that

$$\langle (A, P, y_n, g_n, n), (w_1, w_2, w_3, w_4, w_5)\rangle \in \Psi \iff$$
$$y_n = w_1, w_4 = P(n), w_5 = A(n) \text{ and}$$
$$(\forall m \in \omega)(w_2(m) = A(g_n(m)) \wedge w_3(m) = P(g_n(m)).$$

So Ψ is a Δ^1_1 map and condition (c) describes that $(A, P, y_n, g_n, n) \in \Psi^{-1}(F'')$ thus it defines a $\Pi^1_1(t)$ set. Therefore, using Theorem 5.1.3 we can conclude that F''' is also a $\Pi^1_1(t)$ set.

Now we will prove that F''' contains a "good selection" and then X will be the projection of F''' on the last coordinate.

More precisely, let:

$$x \in X \iff (\exists (z, A, P, p) \leq_h x)((z, A, P, p, x) \in F''').$$

Notice that X is indeed the projection of F''' on the last coordinate: if $(z, A, P, p, x) \in F''' \subset F'$ then $(A, P, p) \leq_h x$ by the definition of F' and from the 3rd point of Lemma 5.2.2 we obtain that $z \leq_h x$, so obviously $(z, A, P, p) \leq_h x$ holds.

Observe that by Theorem 5.1.3 the set X is also $\Pi^1_1(t)$.

Proposition 5.2.8. *For every $\alpha \in \omega_1$ there exists a unique*

$$(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$$

such that $<_{z_\alpha} \cong \alpha$. Moreover,

$$\{A_\alpha(n) : n \in \omega\} = \{x_\beta : \beta < \alpha\}$$

holds for every $\alpha < \omega_1$.

Uniqueness. Let $(z, A, P, p, x), (z', A', P', p', x') \in F'''$ be such that $<_z \cong <_{z'} \cong \alpha$.

$z = z'$: follows form the 1st point of Lemma 5.2.2 since both of $\phi(z, x)$ and $\phi(z', x')$ must hold.

$p = p'$: clear by Remark 5.2.5.

$P = P'$: also from Remark 5.2.5 we have that P and P' are enumerations of the first α reals given by $<_z = <_{z'}$.

$A = A'$: suppose not. Then take the $<_z$ minimal $n \in \omega$ such that $A(n) \neq A'(n)$. By the definition of F''' there exist y_n, g_n and y'_n, g'_n such that $(y_n, A_n, P_n, P(n), A(n)) \in F''$ and $(y'_n, A'_n, P'_n, P'(n), A'(n)) \in F''$, g_n and g'_n are isomorphism between $<_z |_{<_z n}$ and y_n, y'_n and $\phi(y_n, x)$ and $\phi(y'_n, x)$ hold. Then again by Lemma 5.2.2 $y_n = y'_n$, g_n is unique so it must be equal to g'_n. We obtain that $(y_n, A_n, P_n, P(n)) = (y'_n, A'_n, P'_n, P'(n))$ but then $A(n) = A'(n)$ since F' was uniformised.

$x = x'$: also follows from the fact that F' was uniformised.

Existence. Now with transfinite induction we construct for each $\alpha \in \omega_1$ a $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ with the required properties.

Let us formulate the inductive hypothesis: let $\alpha < \omega_1$ be an ordinal and suppose that for every $\beta < \alpha$ we have $(z_\beta, A_\beta, P_\beta, p_\beta, x_\beta) \in F'''$ such that for every $\beta < \alpha$ we have $\{A_\beta(n) : n \in \omega\} = \{x_\gamma : \gamma < \beta\}$.

We will construct $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ satisfying the previous hypothesis.

z_α: using the 1st point of Lemma 5.2.2 there exists a unique z_α such that $<_{z_\alpha} \cong \alpha$ and $(\exists x \in \omega^\omega)\phi(z_\alpha, x)$.

p_α: let p_α be the αth real with respect to $<_L$.

A_α, P_α: The order-preserving bijection between $<_{z_\alpha}$ and α yields enumerations $\{x_\beta : \beta < \alpha\}$ and $\{p_\beta : \beta < \alpha\}$, let $A_\alpha(n)$ be the nth element of the first set's enumeration and define $P_\alpha(n)$ similarly.

By the definition of A_α we have that $\{A_\alpha(n) : n \in \omega\} = \{x_\beta : \beta < \alpha\}$.

We will prove that there exists an $x_\alpha \in \omega^\omega$ such that $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$. By the properties of F for every (A, p) there exist cofinally many (in the hyperdegrees) x such that $(A, p, x) \in F$, so this also holds for (A_α, p_α). From Lemma 5.2.7 we have that if the section $F'_{(z_\alpha, A_\alpha, P_\alpha, p_\alpha)}$ is non-empty then it is cofinal in the hyperdegrees.

Now we show that it is non-empty. $L \models$"P_α is an enumeration of the first α reals given by $<_{z_\alpha}$ and p_α is the αth real" so by absoluteness arguments it holds in $L_{\omega_1^x}$ if ω_1^x is high enough. Let us choose a real x such that $x \in F_{(A_\alpha, p_\alpha)} \cap \mathcal{S}$, $L_{\omega_1^x} \models$"P_α is an enumeration of the first α reals given by $<_{z_\alpha}$ and p_α is the αth real" and $\phi(z_\alpha, x)$. Such an x exists by the 2nd point of Lemma 5.2.2 and by the fact that $F_{(A,p)} \cap \mathcal{S}$ is cofinal in the hyperdegrees. Clearly $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x) \in F'$.

Thus there exists an x_α such that $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F''$.

What remains to show is that $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$:

From $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'$ follows that $\phi(z_\alpha, x_\alpha)$. First notice that by the 4th point of Lemma 5.2.2 $\phi(z_\alpha, x_\alpha)$ implies the existence of y_n-s and g_n-s satisfying properties $2(a)$ and $2(b)$ from the definition of F'''.

To see that $2(c)$ also holds for $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha)$, fix a natural number n. We know that $\phi(y_n, x_\alpha)$ holds thus there exists a $\beta < \alpha$ such that $<_{y_n} \cong \beta$. For all $\beta < \alpha$ the formula $\phi(z_\beta, x_\beta)$ holds (by inductive hypothesis $(z_\beta, A_\beta, P_\beta, p_\beta, x_\beta) \in F''' \subset F'$ and use the 1st point of the definition of F'). Let us set $A_n = A_\alpha \circ g_n$ and $P_n = P_\alpha \circ g_n$.

We will prove that

$$(y_n, A_n, P_n, P_\alpha(n), A_\alpha(n)) = (z_\beta, A_\beta, P_\beta, p_\beta, x_\beta) \in F''.$$

By the 1st property of ϕ the equality $y_n = z_\beta$ holds.

Now using the inductive hypothesis we have that $\{A_\beta(m) : m \in \omega\} = \{x_\gamma : \gamma < \beta\}$. The latter set clearly equals $\{A_n(m) : m \in \omega\}$. A_β and A_n are the enumerations of the same set of reals given by $<_{z_\beta} = <_{y_n}$, hence $A_n = A_\beta$.

Similarly, since P_β and P_n are the enumerations of the same set (namely the β long initial segment of the reals with respect to $<_L$, see the Existence part of the proof and Remark 5.2.5). Finally, $A_\alpha(n)$ and $P_\alpha(n)$ are defined as x_β and the βth real, respectively.

This finishes the proof of the statement that 2(*c*) also holds for the 5-tuple $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha)$ and hence the proof of the existence. □

We have already seen that X is a $\Pi^1_1(t)$ set. Now we check that it is compatible with F. By the previous proposition, for every $\alpha < \omega_1$ there exists a unique element $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F'''$ such that $<_{z_\alpha} \cong \alpha$. This gives us the enumerations $X = \{x_\alpha : \alpha < \omega_1\}$ and $\{p_\alpha : \alpha < \omega_1\}$. Now by 3rd point of the definition of F' we have that if $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F''' \subset F'$ then $L_{\omega_1^{x_\alpha}} \models$"$p_\alpha$ is the αth real with respect to $<_L$" and by absoluteness the same holds in L. Thus we obtain that $\omega^\omega = \{p_\alpha : \alpha < \omega_1\}$. Fix an $\alpha < \omega_1$. By the second claim of Proposition 5.2.8 it is clear that A_α is an enumeration of $\{x_\beta : \beta < \alpha\}$. Furthermore, $(z_\alpha, A_\alpha, P_\alpha, p_\alpha, x_\alpha) \in F''' \subset F'$ thus by the 2nd point of the definition of F' we have that $x_\alpha \in F_{(A_\alpha, p_\alpha)}$, so we can conclude that X is compatible with F. □

5.3 Generalisations and remarks

Now we will prove the following theorem.

Theorem 5.3.1. *($V = L$) Let B be a Borel subset of an arbitrary Polish space, $|B| > \aleph_0$. Suppose that $F \subset (\omega^\omega)^{\leq\omega} \times B \times \omega^\omega$ is a coanalytic set and for all $p \in B$, $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees. Then there exists a coanalytic set $X \subset \omega^\omega$ that is compatible with F.*

Proof. A classical result states that for every uncountable Borel subset B of a Polish space there exists a map $\Psi : \omega^\omega \to B$ that is a Borel isomorphism.

Suppose that F is a set as above. Let us define $G \subset (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$ as follows

$$(A, q, x) \in G \iff (A, \Psi(q), x) \in F.$$

Clearly, G is a coanalytic set thus there exists a $t \in \omega^\omega$ such that $G \in \Pi^1_1(t)$. Of course, each section $G_{(A,q)}$ is cofinal in the hyperdegrees. The direct application of Theorem 5.2.4 yields a $\Pi^1_1(t)$ (therefore coanalytic) set $X \subset \omega^\omega$ that is compatible with G. From the compatibility we obtain the enumeration $\omega^\omega = \{q_\alpha : \alpha < \omega_1\}$. But then $\{\Psi(q_\alpha) : \alpha < \omega_1\}$ is an enumeration of B and clearly, X is compatible with F using this enumeration. □

We can derive an obvious but useful consequence of the previous theorem using that $x \leq_T y$ implies $x \leq_h y$ and omitting the relativisation.

Theorem 5.3.2. *($V = L$) Let P be an uncountable Borel subset of a Polish space. Suppose that $F \subset (\omega^\omega)^{\leq\omega} \times P \times \omega^\omega$ is a coanalytic set and for all $p \in \omega^\omega$, $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the Turing degrees. Then there exists a coanalytic set X that is compatible with F.*

It is also easy to see that in the previous theorem we can replace ω^ω by $\mathbb{R}^n$ or 2^ω etc., since there are recursive Borel isomorphisms between these spaces. Thus we obtain Theorem 5.0.3.

With the same methods one could prove the following strengthening of Theorem 5.2.4:

Theorem 5.3.3. *($V = L$) Let B be a $\Delta^1_1(t)$ subset of ω^ω, $|B| > \aleph_0$. Suppose that $F \subset (\omega^\omega)^{\leq\omega} \times B \times \omega^\omega$ is a $\Pi^1_1(t)$ set and for all $p \in B$, $A \in (\omega^\omega)^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees. Then there exists an $X \in \Pi^1_1(t)$ that is compatible with F.*

Now we will examine the necessity of $(V = L)$.

Theorem 5.3.4. *If the conclusion of Theorem 5.2.4 holds then there exists a Σ^1_2 well-ordering of the reals. In particular, every real is constructible.*

Proof. Fix recursive Δ^1_1 bijections $\Psi_1 : \omega^\omega \to (\omega^\omega)^{\leq\omega} \times \omega^\omega$ and $\Psi_2 : \omega^\omega \to \omega^\omega \times \omega^\omega$.

Let us define the set $F \subset (\omega^\omega)^{\leq\omega} \times \omega^\omega \times \omega^\omega$ as follows:

$(A, p, x) \in F \iff (A, p) = \Psi_1(\pi_1(\Psi_2(x)) \wedge (\forall n)(A(n) \neq x)$,

where π_1 is the projection of $\omega^\omega \times \omega^\omega$ on the first coordinate. So basically x is coding the previous choices and the parameter in the odd coordinates.

F is clearly Δ^1_1. Now for an arbitrary pair (A, p) and $y \in \omega^\omega$ there exist cofinally many $x \in \omega^\omega$ such that $(A, p) = \Psi_1(\pi_1(\Psi_2(x))$ and $y \leq_h x$, hence every section $F_{(A,p)}$ is cofinal in the hyperdegrees. Thus by our hypothesis there exists a Π^1_1 set $X = \{x_\alpha : \alpha < \omega_1\}$ and an enumeration $\omega^\omega = \{p_\alpha : \alpha < \omega_1\}$ such that for every $\alpha < \omega_1$ we have $x_\alpha \in F_{(A_\alpha, p_\alpha)}$, where A_α is an enumeration of $\{x_\beta : \beta < \alpha\}$.

We will define the well-ordering of ω^ω with the help of the given enumeration of X. Since every x_α codes the appropriate p_α, we can order ω^ω by the first appearance of a real p.

Now for $p, q \in \omega^\omega$ let $(p, q) \in E \iff \exists x, y, A, B$

(1) $x, y \in X$, $x \neq y$, $(A, p, x) \in F, (B, q, y) \in F$

(2) $(\forall m)(\forall C)((C, p, A(m)) \notin F \wedge (C, q, B(m)) \notin F)$

(3) $(\exists n)(x = B(n))$.

Since F is Δ^1_1, we have that E is a Σ^1_2 relation.

Fix $p, q \in \omega^\omega$. There exist minimal ordinals α, β such that $p_\alpha = p$ and $p_\beta = q$. We will prove that $(p, q) \in E \iff \alpha < \beta$. We have for α and β that $(A_\alpha, p_\alpha, x_\alpha) \in F$ and $(A_\beta, p_\beta, x_\beta) \in F$.

First, if $\alpha < \beta$ choose $x = x_\alpha$, $y = x_\beta$, $A = A_\alpha$, $B = A_\beta$. Then (1) is obvious (by the definition of F we have that $x_\alpha \neq x_\beta$ if $\alpha < \beta$) and A_β is an enumeration of $\{x_\gamma : \gamma < \beta\}$ so (3) also holds. Suppose that (2) fails for p: there exists a pair m, C

such that $(C, p, A(m)) \in F$ (the other case is similar). Then $A(m) = x_\gamma$ for some $\gamma < \alpha$ and $(C, p) = (A_\gamma, p_\gamma)$. This would contradict the minimality of α, and similarly for β.

For the other direction suppose that $(p, q) \in E$ and take x, y, A, B witnessing this fact. Clearly, $x = x_{\alpha'}$ for some α' so $(A_{\alpha'}, p_{\alpha'}) = (A, p)$ and similarly $(A_{\beta'}, p_{\beta'}) = (B, q)$. Using (2) we get the minimality of α' and β' so they must be equal to α and β.

Suppose that $\alpha \geq \beta$, then of course $\alpha > \beta$. By (3) we have that there exists an $n \in \omega$ such that

$$A_\beta(n) = A_{\beta'}(n) = B(n) = x = x_{\alpha'} = x_\alpha.$$

By the assumption $\{x_\gamma : \gamma < \beta\} \subsetneq \{x_\gamma : \gamma < \alpha\}$. We have that

$$\{A_\beta(m) : m \in \omega\} = \{x_\gamma : \gamma < \beta\} \subset \{A_\alpha(m) :\in \omega\}$$

then $A_\alpha(m) = x_\alpha$ for some $m \in \omega$. But this is a contradiction, since $(\forall n)(A(n) \neq x)$ for every $(A, p, x) \in F$. Thus $\alpha < \beta$.

So we obtain that E is a Σ^1_2 well-ordering. The second claim follows from Mansfield's theorem, see [38, Theorem 25.39]. □

Next we show that the definability assumption on our "selection algorithm" F cannot be dropped in Theorem 5.2.4.

Example 5.3.5. (CH) There exists a family $\{A_\alpha : \alpha < \omega_1\} \subset [\omega^\omega]^{\leq\aleph_0}$ such that if for a set X there exists an enumeration $X = \{x_\alpha : \alpha < \omega_1\}$ such that $(\forall\alpha < \omega_1)(x_\alpha \notin A_\alpha)$ then X is not coanalytic.

Proof. Fix an enumeration of the reals $\{y_\alpha : \alpha < \omega_1\}$. We will define A_α by recursion. Suppose that we are ready for $\beta < \alpha$ and let us choose $A_\alpha \in [\omega^\omega]^{\leq\aleph_0}$ such that for every uncountable $P \in \bigcup_{\beta\leq\alpha} \Pi^1_1(y_\beta)$ we have $|P \cap (A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta)| \geq 2$ and $\bigcup_{\beta<\alpha} A_\beta \subset A_\alpha$ and $y_\alpha \in A_\alpha$. Since $|\bigcup_{\beta<\alpha} A_\beta| \leq \aleph_0$ and $\bigcup_{\beta\leq\alpha} \Pi^1_1(y_\beta)$ is countable, there exists such an A_α.

Now suppose that $X = \{x_\alpha : \alpha < \omega_1\}$ is coanalytic and for every α we have $x_\alpha \notin A_\alpha$. Clearly, $\bigcup_\alpha A_\alpha = \omega^\omega$, thus X must be uncountable. Since X is coanalytic, we have that there exist an α_0 such that $X \in \Pi^1_1(y_{\alpha_0})$. Thus for every $\alpha \geq \alpha_0$ by the construction of A_α's $|X \cap (A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta)| \geq 2$. Now consider the map ϕ that assigns to each $\alpha \geq \alpha_0$ the minimal index $\phi(\alpha)$ such that $x_{\phi(\alpha)} \in A_{\alpha+1} \setminus A_\alpha$. There are at least two distinct elements of X in $A_{\alpha+1} \setminus A_\alpha$ and $x_\gamma \notin A_{\alpha+1}$ for $\gamma > \alpha$ (the constructed family is increasing), hence $\phi(\alpha) < \alpha$. Moreover, ϕ is clearly injective. Therefore, we have that ϕ is a regressive function whose domain is a co-countable subset of ω_1. This contradicts Fodor's lemma. □

Remark 5.3.6. The same holds for any projective class.

Now we will prove a general technical theorem which implies the existence of Π^1_1 Hamel basis, but could be used to prove the existence of Π^1_1 n-point sets, analogous versions for circles, etc. The situation in the following definition is that we have a relation $R(x, y)$

on finite subsets of the reals that intuitively means that x is "stronger" than y in some sense (e.g. in case of Hamel basis all elements of y are linearly generated by x, in case of two-point sets all lines that intersect y in at least two points intersect x in at least two points etc.). Our goal is to find an R-independent set (all the relations are trivial) that is "stronger" than all the finite subsets of the reals. H_B^R will be the set of finite sets that can be added to B preserving it's independence.

Definition 5.3.7. Let R be a binary relation on the finite subsets of $\mathbb{R}^n$.

- We say that a set $X \subset \mathbb{R}^n$ is *R-independent* if for all $x, y \in [X]^{<\omega}$ $R(x, y) \Rightarrow y \subset x$.
- Fix a $k \in \omega$, if for every $y \in [\mathbb{R}^n]^k$ there exists an element $x \in [X]^{<\omega}$ such that $R(x, y)$ then we say that X is a *k-generator set for R.*
- If B is an R-independent set let us use the notation $H_B^R = \{x \in [\mathbb{R}^n]^{<\omega} : x \cup B$ is R-independent$\}$.

We use parameters n and k even though they will not be needed for the proof of the Hamel basis case.

Definition 5.3.8. We will use the following notation: $x \equiv_h y \iff (x \leq_h y \wedge y \leq_h x)$.

The extra difficulty in the construction of a Hamel basis is that in a step we have to put more than one real into our set, so we have to deal with finite sequences. Moreover, to use our method one have to choose reals which are high enough in $\leq_h$. Thus our strategy is to select $\leq_h$ equivalent reals in every step of the procedure.

Definition 5.3.9. Let us denote by $\mathcal{E}$ the set

$$\{x \in [\mathbb{R}^n]^{<\omega} : (\forall x_1, x_2 \in x)(x_1 \equiv_h x_2)\}.$$

Theorem 5.3.10. *($V = L$) Let $t \in \mathbb{R}$ and $n, k \in \omega$ be arbitrary. Suppose that $R \subset [\mathbb{R}^n]^{<\omega} \times [\mathbb{R}^n]^{<\omega}$ is a $\Delta_1^1(t)$ relation that satisfies the property (*):*

for every countable $B \subset \mathbb{R}^n$ the set $\mathcal{E} \cap H_B^R$ is cofinal in the hyperdegrees and if for $y \in [\mathbb{R}^n]^k$ there is no $z \in [B]^{<\omega}$ such that $R(z, y)$ then $\{x : R(x, y)\} \cap \mathcal{E} \cap H_B^R$ is cofinal in the hyperdegrees.

Then there exists an uncountable $\Pi_1^1(t)$, R-independent set that is a k-generator for R.

Proof. Let us define the set $F \subset ([\mathbb{R}^n]^{<\omega})^{\leq\omega} \times \mathbb{R} \times [\mathbb{R}^n]^{<\omega}$ and fix a recursive Borel isomorphism $\Phi : \mathbb{R} \to [\mathbb{R}^n]^k$. $(A, p, x) \in F \iff$
EITHER the conjunction of the following clauses holds

(1) $\bigcup ran(A)$ is R-independent

(2) $(\forall z \in ran(A))(\neg R(z, \Phi(p)))$

(3) $R(x, \Phi(p))$ holds and $x \in \mathcal{E} \cap H^R_{\bigcup ran(A)}$

OR (1) $\wedge \neg$(2) holds and $x \in \mathcal{E} \cap H^R_{\bigcup ran(A)}$
OR $\neg$(1).

Since A is countable and the relation $\equiv_h$ is Π^1_1, we get that F is $\Pi^1_1(t)$. By property (*) every section $F_{(A,p)}$ is cofinal in the hyperdegrees (if $\neg$(1) then this is obvious and the cases when (1) $\wedge \neg$(2) or (1) $\wedge$ (2) holds are exactly described by property (*)) so we can apply Theorem 5.2.4. This gives us a $\Pi^1_1(t)$ set $Y \subset [\mathbb{R}^n]^{<\omega}$ such that $\bigcup ran(Y)$ is R-independent and for every $y \in [\mathbb{R}^n]^k$ there exists an $x \in Y$ such that $R(x, y)$ thus $\bigcup ran(Y)$ is a k-generator for R. Moreover $ran(Y) \subset \mathcal{E}$. Hence it suffices to prove that $X = \bigcup ran(Y)$ is a $\Pi^1_1(t)$ set. But using that for every $x \in Y$ the elements of x are equivalent in hyperdegrees we get

$$a \in X \iff (\exists l \in \omega)(\exists a_1, \dots a_l \leq_h a)(\{a, a_1 \dots a_l\} \in ran(Y)).$$

Applying Theorem 5.1.3 we can verify that $X \in \Pi^1_1(t)$. □

Corollary 5.3.11. *($V = L$) There exists a Π^1_1 Hamel basis.*

Proof. Let us define the relation $R \subset [\mathbb{R}]^{<\omega} \times [\mathbb{R}]^{<\omega}$. $R(x, y) \iff (y \subset \langle x \rangle_{\mathbb{Q}})$ i. e. every element of y is in the linear subspace generated by the elements of x over the rationals. Notice that R is Δ^1_1. In the terminology of the previous theorem X is a Hamel basis if it is R-independent and 1-generator for R. So we just have to check whether property (*) holds.

First, if B is a countable linearly independent subset of the reals then for all but countably many finite sets $a \in [\mathbb{R}]^{<\omega}$ we have $a \in H^R_B$. Therefore obviously H^R_B is cofinal in the hyperdegrees. So the first part of (*) holds.

Now fix an element $y \in \mathbb{R}$, a countable $B \subset \mathbb{R}$ such that there is no $z \in [B]^{<\omega}$ such that $R(z, \{y\})$. We will prove that for every $s \in \mathbb{R}$ there exists a pair $w_1, w_2 \in \mathbb{R}$ satisfying $y = w_1 + w_2$, $w_1 \equiv_h w_2$, $B \cup \{w_1, w_2\}$ linearly independent and $s \leq_h w_1, w_2$. This fact indeed implies that the set $\{x : x \in \mathcal{E} \wedge R(x, y)\} \cap H^R_B$ is cofinal in the hyperdegrees, so the second part of (*) also holds.

Here we repeat Miller's argument. Without loss of generality we can suppose that $y \leq_h s$ and s is not hyperarithmetic in any finite subset of $B \cup \{y\}$ because we can replace s by a more complicated real. We can choose w_1 and w_2 such that s is coded in w_1's odd and w_2's even digits such that $w_1 + w_2 = y$. Then $s \leq_h w_1, w_2$ hence $y \leq_h w_1, w_2$. But then $y = w_1 + w_2$ implies $w_1 \equiv_h w_2$. If $w_1 \in \langle B, w_2 \rangle_{\mathbb{Q}}$ then $y \in \langle B, w_2 \rangle_{\mathbb{Q}} \setminus \langle B \rangle_{\mathbb{Q}}$ and then $w_2 \in \langle B, y \rangle_{\mathbb{Q}}$ but this would imply that s is hyperarithmetic in a finite subset of $B \cup \{y\}$ which is a contradiction. Thus w_1 and w_2 are the appropriate reals.

Thus property (*) holds indeed, and the direct application of Theorem 5.3.10 hence produces a Π^1_1 Hamel basis. □

Finally we will prove another variant of our theorem, considering the case where the choice at step α does not depend on the previous choices.

Theorem 5.3.12. *($V = L$) Let be $t \in \mathbb{R}$ and suppose that $G \subset \mathbb{R}^n \times \mathbb{R}$ is a $\Delta^1_1(t)$ set and for every countable $A \subset \mathbb{R}$ the complement of the set $\cup_{p \in A} G_p$ is cofinal in the hyperdegrees. Then there exists an uncountable $\Pi^1_1(t)$ set $X \subset \mathbb{R}^n$ that intersects every G_p in a countable set.*

Proof. Using Theorem 5.1.4 there exists a Σ_1 formula θ such that

$$a \in G^c \iff L_{\omega_1^{(a,t)}}[a,t] \models \theta(a,t).$$

Now let us define the set H as follows:

$$(x,p) \in H \iff x \in \mathcal{S} \wedge p,t \leq_h x \wedge L_{\omega_1^x} \models (\forall p' \leq_L p)(\theta((x,p'),t)).$$

H is a $\Pi^1_1(t)$ set, for this just repeat the usual argument, that is, $x \in \mathcal{S} \wedge p,t \leq_h x$ implies that $L_{\omega_1^x} = L_{\omega_1^{((x,p),t)}}[((x,p),t)]$ and use Theorems 5.1.4, 5.1.1, 5.1.2 and Lemma 5.2.1. Observe that for a real p

$$H_p = (\bigcap_{p' \leq_L p} G^c_{p'}) \cap \mathcal{S} \cap \{z : p,t \leq_h z\}.$$

Thus, the theorem's conditions imply that for every real p the section H_p is cofinal in the hyperdegrees.

Define $F \subset (\mathbb{R}^n)^{\leq \omega} \times \mathbb{R} \times \mathbb{R}^n$: $(A,p,x) \in F \iff (x,p) \in H \wedge x \notin A$. Obviously for every (A,p) the section $F_{(A,p)}$ is cofinal in the hyperdegrees and F is $\Pi^1_1(t)$. Theorem 5.2.4 provides an uncountable $\Pi^1_1(t)$ set $X \subset \mathbb{R}^n$ and enumerations $X = \{x_\alpha : \alpha < \omega_1\}$, $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$ and an enumeration A_α (in type $\leq \omega$) of $\{x_\beta : \beta < \alpha\}$ such that $x_\alpha \in F_{(A_\alpha, p_\alpha)} = H_{p_\alpha} \setminus \{x_\beta : \beta < \alpha\}$. Suppose that there exists a $p \in \mathbb{R}$ for which $|X \cap G_p| > \aleph_0$. Then $p_\beta >_L p$ if β is high enough, since only countably many p_α's are $<_L$ less then p. But if $p_\beta >_L p$ then $x_\beta \in G^c_p$. □

Now Theorem 5.0.4 is a trivial consequence of Theorem 5.3.12.

5.4 Applications

Theorem 5.0.3 can be applied in various situations. Let us remark here that one can obtain Π^1_1 sets instead of coanalytic ones by just repeating the proofs and using Theorem 5.2.4 in all the theorems of this section. We will prove the simpler (boldface) versions for the sake of transparency.

Theorem 5.4.1. *($V = L$) There exists a coanalytic MAD family.*

Proof. First fix a recursive partition $B = \{B_i : i \in \omega\}$ of ω to infinite sets. Define $F \subset (\mathcal{P}(\omega))^{\leq \omega} \times \mathcal{P}(\omega) \times \mathcal{P}(\omega)$ as follows: $(A,p,x) \in F \iff$
EITHER the conjunction of the following clauses holds

(1) $ran(A) \cup B$ contains pairwise almost disjoint elements

(2) p is almost disjoint form the elements of $ran(A) \cup B$

(3) $p \subset x$ and x is almost disjoint form the elements of $ran(A) \cup B$

OR (1) $\wedge \neg$(2) holds and x is almost disjoint form the elements of $ran(A) \cup B$
OR $\neg$(1).

Clearly, F is Borel. What we have to prove is that for all pairs (A, p) the section $F_{(A,p)}$ is cofinal in the Turing degrees.

Suppose that (1) and (2) hold, let $u \in \mathcal{P}(\omega)$ be an arbitrary real. Choose $x' = p \cup \bigcup_{i \in \omega} F_i$, where $F_i \subset B_i$ are finite and if $i > j$ then $A(j) \cap F_i = \emptyset$ and

$$|(p \cup F_i) \cap B_i| \equiv 1 \mod 2 \iff u(i) = 1.$$

For every i there exist such an F_i, since the B_i's are disjoint and infinite, and $ran(A) \cup B$ contains pairwise almost disjoint sets. Then x' satisfies 3 and $u \leq_T x'$.

Now in the case when (1) $\wedge \neg$(2) holds our job is easier: e. g. we can repeat the previous argument omitting p.

Finally, if $\neg$(1) is true then $F_{(A,p)} = \mathcal{P}(\omega)$.

Notice that Theorem 5.0.3 was stated in the form that the set of the parameters is $\mathbb{R}$ but we can easily replace it by $\mathcal{P}(\omega)$ using a recursive Borel isomorphism.

So we can apply Theorem 5.0.3 and we get a coanalytic set $X = \{x_\alpha : \alpha \in \omega_1\}$ such that X is compatible with F. It is obvious by transfinite induction that the elements of X are pairwise almost disjoint. It is also clear that $X \cup B$ is maximal since for every real p there exists an $\alpha < \omega_1$ such that $p_\alpha = p$. Thus, there exists an element of X that is not almost disjoint from p. $\square$

Theorem 5.4.2. *($V = L$) There exists a coanalytic two-point set.*

Proof. For each real $p \in \mathbb{R}$ fix a line l_p such that it is the line defined by the equation $((p)_1)x + ((p)_2)y = (p)_3$, where $(p)_1, (p)_2$ and $(p)_3$ are the reals made of every $3k$th, $3k+1$st and $3k+2$nd digit of p. l_p can be empty, however every line appears at least two times. Let us define $F \subset (\mathbb{R}^2)^{\leq\omega} \times \mathbb{R} \times \mathbb{R}^2$ by $(A, p, x) \in F \iff$
EITHER the conjunction of the following clauses holds

(1) there are no 3 collinear points in $ran(A)$

(2) $|ran(A) \cap l_p| < 2$ and $l_p \neq \emptyset$

(3) $x \in l_p \setminus ran(A)$ and x is not collinear with any two distinct points of $ran(A)$

OR (1) $\wedge \neg$(2) holds and x is not collinear with two distinct points of $ran(A)$
OR $\neg$(1).

Now F is clearly Borel. What we have to check is that for all (A, p) the section $F_{(A,p)}$ is cofinal in the Turing degrees. Fix a pair (A, p). If $(1) \wedge (2)$ holds then the section is equal to l_p minus a countable set. Every line is cofinal in the Turing degrees, because we can choose one of the coordinates arbitrarily. Now notice that if H is a set which is cofinal in the Turing degrees and H' is countable the $H \setminus H'$ is still cofinal: to see this let u be an arbitrary real and let s be such that $(\forall s' \in H')(s' \not\geq_T s)$ then there exist $r \in H$ such that $s, u \leq_T r$ and clearly $r \notin H'$. So we have that if $(1) \wedge (2)$ holds then $F_{(A,p)}$ is cofinal in the Turing degrees.

If $(1) \wedge \neg(2)$ holds then we just have to choose an arbitrary point that is not collinear with any two distinct points of A. The case when (1) is false is obvious.

Thus, by Theorem 5.0.3 we get an uncountable coanalytic set $X = \{x_\alpha : \alpha < \omega_1\} \subset \mathbb{R}^2$. One can easily verify that X cannot contain three collinear points. Moreover, since every line l_p appears at least twice, $|l_p \cap X| = 2$. □

Similar statements can be formulated for n-point sets, circles, appropriate algebraic curves etc., the above method works in these cases.

5.4.1 Curves in the plane

Now we will consider the following question: What can we say about a set in the plane which intersects every "nice" curve in a countable set? Let us call a continuously differentiable $\mathbb{R} \to \mathbb{R}^2$ function a C^1 curve.

Definition 5.4.3. We say that a set $H \subset \mathbb{R}^2$ is C^1*-small* if the intersection of H with the range of every C^1 curve is a countable set.

In [33] the authors proved that assuming Martin's axiom and the Semi-Open Coloring Axiom if H is C^1-small then $|H| \leq \aleph_0$. Moreover, they showed in ZFC that no perfect set is C^1-small. Thus, no uncountable analytic set is C^1-small. On the other hand, the following proposition holds.

Proposition 5.4.4. *(CH) There exists an uncountable C^1-small set.*

Proof. We will prove later that the union of the range of countably many C^1 curves cannot cover the plane. This implies the statement by an easy transfinite induction. □

Thus, it is an interesting question whether an uncountable C^1-small subset can be coanalytic. We will apply Theorem 5.0.4.

Theorem 5.4.5. *$(V = L)$ There exists an uncountable C^1-small coanalytic set.*

Proof. First we have to prove that there exists a Borel set $G \subset \mathbb{R}^2 \times \mathbb{R}$ such that if γ is a C^1 curve then there exists a $p \in \mathbb{R}$ such that $G_p = ran(\gamma)$.

One can easily prove that the set B of C^1 curves as a subset of $C(\mathbb{R}, \mathbb{R}^2)$ is a Borel set (see e.g. [40, 23. D]). The set $\{((x,y),\gamma) : (x,y) \in ran(\gamma)\} \subset \mathbb{R}^2 \times C(\mathbb{R},\mathbb{R}^2)$ is clearly closed. So $(\mathbb{R}^2 \times B) \cap \{((x,y),\gamma) : (x,y) \in ran(\gamma)\}$ is also a Borel set. Furthermore, there exists a Borel isomorphism $\phi : \mathbb{R} \to B$ since these two are standard Borel spaces of cardinality $\mathfrak{c}$ and we can apply the isomorphism theorem. Now we can define $G \subset \mathbb{R}^2 \times \mathbb{R}$: $((x,y),p) \in G \iff ((x,y),\phi(p)) \in (\mathbb{R}^2 \times B) \cap \{((x,y),\gamma) : (x,y) \in ran(\gamma)\}$ which is a Borel set and for every $\gamma \in C^1$ there exists a $p \in \mathbb{R}$ such that $G_p = ran(\gamma)$.

To apply Theorem 5.0.4 we have to check that if we have countably many C^1 curves $\{\gamma_i : i \in \omega\}$ then the complement of the union of their ranges is cofinal in the Turing degrees. For this it is enough that there exists a line l such that

$$|l \cap \bigcup(\{ran(\gamma_i) : i \in \omega\})| \leq \aleph_0.$$

Let us concentrate only on the horizontal lines. For a curve γ_i take let $f_i(x) = \pi_y(\gamma_i(x))$, i. e. the composition with the projection on the vertical axis. f_i is C^1 function, thus by Sard's lemma the set $H_i = \{y \in \mathbb{R} : (\exists x)(f_i'(x) = 0 \wedge f_i(x) = y\}$ has Lebesgue measure zero. Let $b \in \mathbb{R} \setminus (\cup H_i)$. Then the line $\{(x,b) : x \in \mathbb{R}\}$ intersects every curve γ_i in countably many points, since otherwise it would be an image of a critical value.

Finally, the application of Theorem 5.0.4 produces an uncountable C^1-small coanalytic set. □

5.5 Open problems

In Theorem 5.0.3 the set of the parameters is a Borel set and this was used in the proof numerous times.

Question 5.5.1. *Does Theorem 5.0.3 hold if we only assume that B is coanalytic?*

As a partial converse we have proved that the conclusion of Theorem 5.2.4 implies that every real constructible. It is natural to ask whether the converse also holds.

Question 5.5.2. *Does the conclusion of Theorem 5.0.3 hold if every real is constructible?*

One of the weaknesses of the method is that the constructed set X is a subset of $\mathcal{S}$. It is known (see e. g. [41]) that $\mathcal{S}$ is the largest thin (not containing a perfect subset) Π^1_1 set. Thus non of the constructed sets contain a perfect subset. In the case of C^1-small sets this cannot be expected, but how about the other constructions?

Question 5.5.3. *Is it consistent that there exists a Π^1_1 Hamel basis (two-point set, MAD family) that contains a perfect subset?*

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Summary

The thesis is concerned with four problems of descriptive set theory.

In Chapter 2 we investigate the properties of negligible sets in Polish topological groups. Answering questions of D. Fremlin and J. Mycielski we prove that, contrary to the case of locally compact Polish groups, in the non-locally compact case Haar null sets do not possess certain regularity properties. In particular, we show that in every abelian non-locally compact Polish group there exists a Borel Haar null set that does not have a G_δ Haar null hull.

Chapter 3 tackles the problem of characterisation of the linearly ordered sets of Baire class 1 functions on Polish spaces ordered by the pointwise ordering. Solving a problem that was posed by M. Laczkovich in the 70s we give a full characterisation of such linearly ordered sets in terms of a universal linearly ordered set. Namely, there exists a concrete, combinatorially definable linearly ordered set such that a linearly ordered set is order isomorphic to a linearly ordered set of Baire class 1 functions if and only if it can be embedded order preservingly into our universal linearly ordered set. Using this result, we easily reprove the theorems of K. Kuratowski, P. Komjáth, M. Elekes and J. Steprāns and we answer all of the known open questions concerning the linearly ordered sets of Baire class 1 functions. The results of Chapter 2 and 3 are joint work with M. Elekes.

A. Kechris and A. Louveau built an extensive theory of ranks defined on the first Baire class. However, this has no straightforward generalisation to Baire class ξ when $\xi \geq 2$. Chapter 4 deals with defining well behaved ranks (e.g. subadditive) on Baire class ξ for $\xi \geq 2$. A direct consequence of the results can be used in solving infinite systems of functional equations. These are joint results with M. Elekes and V. Kiss.

Chapter 5 is devoted to the precise formulation and generalisation of a method discovered by A. W. Miller. Miller proved that under certain assumptions one can inductively construct subsets of Polish spaces with low complexity and nice regularity properties. We precisely formulate a "black box" condition that can be used in such situations without understanding the theories behind Miller's argument. As an application, we reprove Miller's theorems and present some new results.

Magyar nyelvű összefoglaló

A disszertáció négy leíró halmazelméleti problémát dolgoz fel.

A 2. fejezet lengyel topologikus csoportok kis részhalmazainak különféle tulajdonságait vizsgálja. David Fremlin és Jan Mycielski kérdéseit megválaszolva belátjuk, hogy nem lokálisan kompakt lengyel csoportokban a Haar-null halmazok nem rendelkeznek bizonyos, a lokálisan kompakt esetben könnyen igazolható regularitási tulajdonsággal. Fő eredményünk, hogy minden Abel nem lokálisan kompakt csoportban létezik olyan Haar-null Borel halmaz, amelynek nincs G_δ Haar-null burka.

A disszertáció 3. fejezete a Baire 1 függvények rendezett halmazait vizsgálja a pontonkénti rendezés szerint. Laczkovich Miklós egy, a 70-es években kitűzött problémáját megoldva megmutatjuk, hogy létezik egy kombinatorikusan leírható univerzális rendezett halmaz, azaz egy olyan rendezett halmaz, hogy egy rendezett halmaz pontosan akkor áll elő mint Baire 1 függvények lineárisan rendezett halmaza, ha az rendezéstartóan beágyazható az univerzális rendezett halmazba. Ezt az eredményt felhasználva könnyen újrabizonyítjuk Kazimierz Kuratowski, Komjáth Péter, Elekes Márton és Juris Steprāns tételeit. Ezen felül megválaszolunk minden, a Baire 1 függvények lineárisan rendezett részhalmazaival kapcsolatos ismert nyitott kérdést. Az első két fejezet Elekes Márton és a szerző közös eredményeit tartalmazza.

A rangfüggvények elméletét a Baire 1 függvényosztályon Alexander S. Kechris és Alain Louveau építették ki. Meglepő módon eredményeiknek nincsen kézenfekvő általánosítása a magasabb Baire-osztályokra. A 4. fejezet témája jól viselkedő rangok általánosítása ezekre az osztályokra. Elekes Mártonnal és Kiss Viktorral közös eredményeink közvetlen következményeként adódik egy tétel végtelen függvényegyenlet-rendszerek megoldhatóságával kapcsolatban.

Az 5. fejezet egy Arnold W. Miller által felfedezett módszert általánosít. A transzfinit rekurzióval konstruált halmazok általában nem definiálhatóak, mérhetőek vagy Baire-tulajdonságúak. A Miller-módszer lényege, hogy bizonyos feltételek mellett mégis elérhető, hogy a kapott halmazok szép tulajdonságokkal rendelkezzenek. A fejezetben precízen megfogalmazunk egy feltételt, amely összegzi Miller technikáját, és felhasználható a technika mögötti elmélet ismerete nélkül. Ezt alkalmazva újrabizonyítjuk Miller tételeit és belátunk néhány új állítást is.